

**The Diffeomorphism Group Approach to Vorticity Model  
Equations**

by

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The Diffeomorphism Group Approach to Vorticity Model Equations

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### **Abstract**

In fluid mechanics, the vorticity provides a valuable alternative perspective of the behavior of flows. Constantin-Lax-Majda [10] approached studying the 3D vorticity equation by proposing a 1D model equation with significant analytic similarities, the Constantin-Lax-Majda equation. This has been followed by a collection of model equations in both 1D and 2D whose behaviors capture many aspects of the full 3D equations.

This thesis contains many new results for several of these equations. We begin by outlining the original analytic theory as well as the Euler-Arnold theory which studies these equations as geodesic equations on infinite dimensional manifolds. We build on the work of Castro-Córdoba [8] and Bauer-Kolev-Preston [4] to show that every solution to the Wunsch equation, a special case of the generalized Constantin-Lax-Majda equation, blows up in finite time. This result also applies to the Constantin-Lax-Majda equation itself. We also investigate the Euler-Weil-Petersson equation which has significant links to the Wunsch equation in the context of Teichmüller theory.

Additionally, we lay the foundations for a geometric theory of the surface quasi-geostrophic equation (SQG). Originally discovered in the context of geophysical fluid mechanics (see Pedlosky [61]), SQG was proposed by Constantin-Majda-Tabak [12] as a 2D version of the 1D Constantin-Lax-Majda equation. In a blog post, Tao [76] showed that SQG arises as the critical point of a functional. This discovery naturally leads to the formulation of SQG as an Euler-Arnold equation. In this thesis we show that the associated geometric space has a smooth, non-Fredholm Riemannian exponential map, and has arbitrarily large curvature of both signs.

Finally we discuss the geometric setting for the Axi-symmetric Euler equations. Here we consider a 3D analogue of the 2D flows considered in Preston [63]. Surprisingly, while the 2D flows

exhibit negative curvature, we show that the corresponding 3D flows exhibit positive curvature and a rich structure of conjugate points. Such a result may have significant ramifications for our understanding of the nature of stability in 2D and 3D fluids.

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## Preface

The bulk of the new results in this dissertation are reproduced from Preston-Washabaugh [67], Washabaugh [81], and Washabaugh-Preston [82]. This thesis aims to:

- Put much of the foundational material for the above papers in one place. The preliminaries section to this thesis can be thought of as a road map containing all the necessary material and papers to start reading Preston-Washabaugh [67], Washabaugh [81], and Washabaugh-Preston [82], as long as one has the basic requirements discussed below.
- Discuss some results not in these papers that are interesting and relevant, but have unclear ramifications, such as the simulations of conformal welding in section 2.2 and the discussion of the possibility of conjugate cascades in section 2.4.
- Place the papers and their results in a broader context with many open problems, some of which are ripe for attack, and others of which are likely far off into the future.

This thesis assumes familiarity with basic graduate level Analysis, Differential and Riemannian Geometry, as well as PDEs, although some of the relevant material from these disciplines will be covered in the introduction.

## Chapter 1

### Introduction

#### 1.1 Euler and Vorticity Model Equations

The motivations for this thesis come from a diverse collection of ideas, perspectives, and methodologies. In this thesis we'll frequently work on  $\mathbb{R}^n$  for  $n = 2$  or  $3$  and we'll also work on a variety of compact manifolds. The common thread behind it all comes from the Euler equations governing non-viscous incompressible fluids. In the case of an open domain  $U \subset \mathbb{R}^3$  with smooth boundary, they are given by:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p \\ \nabla \cdot u = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where  $u$  is the velocity field for the fluid,  $p$  is the pressure, and  $u_0$  is a given initial velocity field. The first equation is the momentum equation, the second equation is the condition that  $u$  be divergence free. If one adds the term  $\nu \Delta u$  to the right hand side of the momentum equation, one gets the Navier-Stokes equations for viscous incompressible fluids. There are critical difficulties with these equations that one can immediately note. In particular,  $p$  has not been given; it must be determined from  $u$  by solving an elliptic PDE. Additionally, the transport term  $(u \cdot \nabla)u$  is non-linear. Hence this is a system of non-linear integral-differential equations along with the divergence-free criterion.

These equations have been known and studied for hundreds of years and in innumerable contexts. A good general introduction is the book Majda-Bertozzi [48]. Despite all this, fundamental

features of them are not known. In particular, while it is known that solutions to these equations exist locally in time in a variety of function spaces, it is currently not generally known if they exist for all time. That is, an initially smooth velocity field could lose its differentiability properties after a finite period of time. Solving this problem could have such profound implications for our understanding of viscosity, turbulence, and fluids themselves that for the Navier-Stokes equations it has been designated as a Millennium problem in Fefferman [30] and as such is widely considered to be one of the most important unsolved problems in mathematics. In fact, it has been shown by Constantin [11] that on either  $\mathbb{R}^3$  or  $T^3$  (the 3-torus), the solution to the Euler equations (1.1) exist at least as long as the Navier-Stokes equations for a small enough  $\nu$ . Given the many similarities of the equations, solving the problem for (1.1) would be of monumental significance.

One of the most productive approaches to studying the Euler equations, Navier-Stokes equations, and fluid-like equations in general has been to study the **vorticity** of the flow given by:  $\omega = \nabla \times u$ . Taking curls of (1.1) yields:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u \\ \omega = \nabla \times u \\ \omega(0, x) = \omega_0(x). \end{cases} \quad (1.2)$$

The vorticity is a direct measure of how much rotation there is in the fluid flow. We can recover  $u$  from knowledge of  $\omega$  with the Biot-Savart law:

$$u(x, t) = -\frac{1}{4\pi} \int \frac{(x - y)}{|x - y|^3} \times \omega(y, t) dy. \quad (1.3)$$

Note in particular that we've replaced the non-local problem of finding  $p$  with the non-local problem of translating between  $u$  and  $\omega$ . The term  $u \cdot \nabla \omega$  is known as the **vorticity transport term** and the term  $\omega \cdot \nabla u$  is known as the **vortex stretching term**. For the 2D Euler equations the vorticity

equation appears as:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0 \\ \omega = \nabla \times u \\ \omega(0, x) = \omega_0(x). \end{cases} \quad (1.4)$$

Note the fundamental difference here: there is no stretching term. The equation  $\omega_t + u \cdot \nabla \omega = 0$  in fact indicates that the vorticity is conserved along fluid particle trajectories. The famous paper Beale-Kato-Majda [5] demonstrated the following fundamental result in the case when the domain of the flow is  $\mathbb{R}^3$ , although it applies for every  $n$ :

**Theorem 1** (Beale-Kato-Majda [5]). *Let  $u$  be a solution of the 3D Euler equations in  $C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})$  and suppose there is a time  $T_*$  such that the solution cannot be continued to  $T_*$  in this space. Assume that  $T_*$  is the first such time. Then*

$$\int_0^{T_*} |\omega(t)|_{L^\infty} dt = \infty. \quad (1.5)$$

and in particular

$$\limsup_{t \uparrow T_*} |\omega(t)|_{L^\infty} = \infty, \quad (1.6)$$

where  $s \in \mathbb{R}$ ,  $s \geq 3$ , and  $H^s$  is the Sobolev space with index  $s$ . These properties give one substantial control over the behavior of solutions to (1.4). In particular, it's known that the solutions exist globally in time. Looking back towards equation (1.2), one of the main obstructions to proving global existence turns out to be the vortex stretching term  $\omega \cdot \nabla u$ . As indicated in the title, one of the central concepts of this thesis is the study of **vorticity model equations**. The idea is to mathematically replicate the vorticity equation (1.2) in lower dimensions to study the roles of vorticity transport and stretching in a possibly simpler environment. Note in particular that this is fundamentally different from studying the vorticity equation for the 2D Euler equation.

The study of vorticity model equations began in Constantin-Lax-Majda [10]. The starting

point for their investigation was that the 3D vorticity equation (1.2) can be re-written as

$$\begin{cases} \omega_t + u \cdot \nabla \omega = D(\omega)\omega \\ \omega = \nabla \times u \\ \omega(0, x) = \omega_0(x), \end{cases} \quad (1.7)$$

where  $D = \frac{1}{2}(\nabla u + \nabla u^T)$ , the deformation matrix, is given in terms of  $\omega$  by a principal value integral of  $\omega$  (essentially the Biot-Savart law). Their main idea was to use what is essentially the unique 1D version of  $D$ , employing the Hilbert transform.

### 1.1.1 The Hilbert and Riesz Transforms

The Hilbert transform arises in a variety of ways. The simplest comes from complex analysis. Suppose one is given a real valued function  $f \in C^\infty(S^1)$ . Is there a holomorphic function  $\phi$  on the disk such that the real part of  $\phi$  restricted to the circle is  $f$ ? Part of the solution to this problem comes from the Hilbert transform. Let  $\text{Re}(\phi) = p$  and  $\text{Im}(\phi) = q$ . Since  $\phi$  is holomorphic, we know that  $p$  must satisfy

$$\begin{cases} \Delta p = 0 & \text{in } \mathbb{D} \\ p|_{S^1} = f \end{cases} \quad (1.8)$$

To obtain  $q$ , we employ the polar Cauchy-Riemann equations:

$$\begin{cases} rp_r(r, \theta) = q_\theta(r, \theta) \\ rq_r(r, \theta) = -p_\theta(r, \theta). \end{cases} \quad (1.9)$$

In particular, in Fourier space,

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}, \quad g(\theta) = \sum_{n \in \mathbb{Z}} d_n e^{in\theta}, \quad (1.10)$$

and

$$p(r, \theta) = \sum_{n \in \mathbb{Z}} r^{|n|} c_n e^{in\theta}, \quad q(r, \theta) = \sum_{n \in \mathbb{Z}} r^{|n|} d_n e^{in\theta}, \quad (1.11)$$

hence the first equation of (1.9) restricted to  $S^1$  says that

$$\sum_{n \in \mathbb{Z}} |n| c_n e^{in\theta} = \sum_{n \in \mathbb{Z}} i n d_n e^{in\theta}. \quad (1.12)$$

Equating Fourier coefficients yields  $d_n = -i \operatorname{sign}(n) c_n$ .

**Definition 1.1.1.** *Given a mean-zero real valued function  $f \in C^\infty(S^1)$ , the **Hilbert transform** of  $f$ , denoted by  $Hf$  is the imaginary part restricted to  $S^1$  of the unique holomorphic function in the disk with real part  $f$  on  $S^1$ . In Fourier space this means that*

$$H(f) = H\left(\sum_{n \in \mathbb{Z}} c_n e^{in\theta}\right) = \sum_{n \in \mathbb{Z}} -i \operatorname{sign}(n) c_n e^{in\theta}. \quad (1.13)$$

*Alternatively, one may define the Hilbert transform as a singular integral operator:*

$$Hf(\theta) = \frac{1}{2\pi} P.V. \int_0^{2\pi} f(\psi) \cot\left(\frac{\theta - \psi}{2}\right) d\psi. \quad (1.14)$$

Note that this definition extends to Sobolev spaces (to be discussed later) as well as to functions defined on  $\mathbb{R}$  being extended holomorphically to the upper half plane. In particular, for functions in  $L^2(\mathbb{R})$  the equation (1.14) becomes:

$$Hf(x) = \frac{1}{\pi} P.V. \int \frac{f(x-y)}{y} dy. \quad (1.15)$$

The integral formulation of  $H$  connects us to many major fields of analysis, in particular that of singular integral operators. Of importance to us is the following theorem from Stein ([75], pg. 55):

**Theorem 2** (Stein [75]). *Suppose  $T$  is a bounded operator on  $L^2(\mathbb{R}^1)$  which satisfies the following properties:*

- *$T$  commutes with translations*
- *$T$  commutes with positive dilations*
- *$T$  anti-commutes with the reflection  $f(x) \rightarrow f(-x)$ .*

*Then  $T$  is a constant multiple of the Hilbert transform.*

This theorem singles out the Hilbert transform as essentially the unique operator in 1D with these properties; an important point we'll return to later. For now, a natural question to ask is what the higher dimensional version of the Hilbert transform might be. We now define:

**Definition 1.1.2.** *On an  $n$ -dimensional manifold  $M$ , the **Riesz transform** is a map from functions to vector fields given by:*

$$\mathcal{R}f = \nabla(-\Delta)^{-1/2}f. \quad (1.16)$$

For example, if  $f(x) = \int c(\xi)e^{i\xi x}d\xi$  in Fourier space on  $\mathbb{R}^2$ , the components of the Riesz transform are given by:

$$(\mathcal{R}f(x))_j = \int i \frac{\xi_j}{|\xi|} c(\xi)e^{i\xi x}d\xi. \quad (1.17)$$

On  $\mathbb{R}^2$  we may also express the Riesz transform in terms of its integral formulation:

$$\mathcal{R}f(x) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{(x-y)}{|x-y|^3} f(y)dy \quad (1.18)$$

As a singular integral operator the Riesz transform really is a higher dimensional version of the Hilbert transform as is indicated in the following theorem from Stein ([75], pg.58):

**Theorem 3** (Stein [75]). *Let  $T = (T_1, T_2, \dots, T_n)$  be an  $n$ -tuple of bounded transformations on  $L^2(\mathbb{R}^n)$ . Suppose*

- *Each  $T_j$  commutes with the translation of  $\mathbb{R}^n$*
- *Each  $T_j$  commutes with the dilations of  $\mathbb{R}^n$*
- *For every rotation  $\rho = (\rho_{jk})$  of  $\mathbb{R}^n$ ,  $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$ .*

*Then the  $T_j$  are a constant multiple of the Riesz transforms, i.e., there exists a constant  $c$  so that  $T_j = cR_j$ ,  $j = 1, \dots, n$ .*

Additionally, while higher dimensional versions of complex analysis can be quite difficult to deal with (more on this later), the Riesz transform does have an interpretation in terms of the generalized Cauchy-Riemann equations, again this can be found in Stein ([75], pg.65):

**Theorem 4** (Stein [75]). *Let  $f$  and  $f_1, \dots, f_n$  all belong to  $L^2(\mathbb{R}^n)$ , and let their respective Poisson integrals be  $u_0(x, y) = P_y * f$ ,  $u_1(x, y) = P_y * f_1, \dots, u_n(x, y) = P_y * f_n$ . Then a necessary and sufficient condition that*

$$f_j = (\mathcal{R}f)_j \quad (1.19)$$

*is that the following generalized Cauchy-Riemann equations hold:*

$$\begin{cases} \sum_{j=0}^n \frac{\partial u_j}{\partial x_j} = 0, \\ \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad j \neq k, \quad \text{with } x_0 = y. \end{cases} \quad (1.20)$$

We will in fact not use the Riesz transform too much. It turns out that a much better model for the deformation matrix is given by a very closely related operator:

**Definition 1.1.3.** *The **Perpendicular Riesz transform** is given by:*

$$\mathcal{R}^\perp f = \nabla^\perp (-\Delta)^{-1/2} f. \quad (1.21)$$

In fact, Preston [68] demonstrated some tantalizing connections of the perpendicular Riesz transform to quaternionic analysis, in that it can be connected to solutions of the Cauchy-Riemann-Feuter equations as in the definition of the Hilbert transform and Theorem 4.

### 1.1.2 gCLM and SQG

Going back to Constantin-Lax-Majda [10], we recall from our previous discussion that a 1D analogue to the operator  $D$  in the full 3D vorticity equation (1.7) is the Hilbert transform. By Theorem 2, the Hilbert transform is the unique such operator we should consider. Hence it makes sense when constructing a 1D model of (1.7) one should include it. Keeping the vortex-stretching term and discarding the transport term Constantin-Lax-Majda [10] wrote down what is now known as the Constantin-Lax-Majda equation (CLM):

$$\begin{cases} \frac{\partial \omega}{\partial t} = H(\omega)\omega \\ \omega = H u_x \\ \omega(x, 0) = \omega_0(x). \end{cases} \quad (1.22)$$

**Note:** the Hilbert transform used by Constantin-Lax-Majda [10] is the negative of our Hilbert transform.

They then demonstrated that the solution can be solved explicitly and blows up in finite time:

**Theorem 5** (Constantin-Lax-Majda [10]). *Suppose  $\omega_0(x)$  is a smooth function decaying sufficiently rapidly as  $|x| \rightarrow \infty$ . Then the solution to the CLM equation is given by*

$$\omega(x, t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \quad (1.23)$$

They also showed:

**Theorem 6** (Constantin-Lax-Majda [10]). *The smooth solution to CLM blows up in finite time if and only if the set  $Z$  defined by*

$$Z = \{x | \omega_0(x) = 0 \text{ and } H\omega_0(x) > 0\} \quad (1.24)$$

*is not empty.*

In other words, this among other things indicates that the vortex stretching term in general contributes significantly to the blow-up of solutions. Of course, one also has a transport term in the 3D vorticity equation (1.7). This issue was addressed by De Gregorio [17], in which he added a transport term to obtain the De Gregorio equation:

$$\begin{cases} \frac{\partial \omega}{\partial t} + u\omega_\theta - \omega u_\theta = 0 \\ \omega = Hu_x \\ \omega(x, 0) = \omega_0(x). \end{cases} \quad (1.25)$$

De Gregorio argued that this equation should be a better model of the 3D vorticity equation (1.7). First, it contains a transport term which will give us a closer resemblance to (1.7). Second, it's known that CLM (1.22) with viscosity actually blows up faster than without which is quite the opposite of what should happen. For example, Burger's equation (1.45) blows up in finite time until a viscosity term  $\nu\Delta$  is added to the right side, then it exhibits global existence. Unfortunately

(1.25) is nowhere near as easy to solve and in fact global existence remains a difficult open problem for it to this day.

This then became the starting point for the investigations of Okamoto-Sakajo-Wunsch [58]. In this paper they put forward the following proposed model, which we will consider over the circle,  $S^1$ , rather than  $\mathbb{R}$ . It is referred to as the generalized Constantin-Lax-Majda Equation (gCLM):

$$\begin{cases} \frac{\partial \omega}{\partial t} + au\omega_\theta - \omega u_\theta = 0 \\ \omega = Hu_\theta \\ \omega(\theta, 0) = \omega_0(\theta), \end{cases} \quad (1.26)$$

where the transport term is given a weight  $a$ . For our purposes it will be more convenient to rescale  $\theta$  by  $b = -1/a$  we obtain:

$$\begin{cases} \frac{\partial \omega}{\partial t} + u\omega_\theta + b\omega u_\theta = 0 \\ \omega = Hu_\theta \\ \omega(\theta, 0) = \omega_0(\theta), \end{cases} \quad (1.27)$$

in other words, we end up weighting the vortex stretching term instead of the transport term. Note that  $b = -1$  corresponds to the De Gregorio equation (1.25). The current list of results and conjectures from Okamoto-Sakajo-Wunsch [58] is:

- For  $b > 0$  it's known the solution to (1.27) may blow up. There are many results in this direction (such as [8] and [4]) which will be discussed later. In fact, in section 2.1.1 of this thesis, we show that when  $b = 2$  **every** mean zero solution blows up. This result also appears to apply to many members of this family, such as the CLM equation itself.
- For  $b = 0$  the solution exists for all time.
- For  $b < 0$  it's generally not known whether the solution blows up or not. It's conjectured that there exists a critical value  $b_* < 0$  such that all solutions of (1.27) exist globally in time for  $0 > b > b_*$  and blowup for  $b < b_*$ . In fact, there has been recent progress made in

Elgindi-Jeong [22] in which the authors demonstrated the existence of a  $b_* < 0$  for which blow-up occurs for  $b < b_*$ .

Overall, the picture that emerges is one in which the strength of the stretching term,  $b\omega u_\theta$ , crucially controls the long-term behavior of the solution. The first part of this thesis, section 2 is an investigation of these issues, mostly focusing on the case  $b = 2$ .

This still leaves the issue of a good 2D model for the 3D vorticity equation (1.7). This question was originally approached in Constantin-Majda-Tabak [12]. In this paper, the authors made the observation that the 2D generalization to the CLM equation (1.22) is the surface quasi-geostrophic equation (SQG):

$$\begin{cases} \nabla^\perp \theta_t + (u \cdot \nabla) \nabla^\perp \theta = \nabla u \cdot \nabla^\perp \theta \\ u = \mathcal{R}^\perp \theta \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (1.28)$$

This form of the equation makes it clear what we may think of as being the transport and stretching terms, however the equation is more commonly written as:

$$\begin{cases} \theta_t + u \cdot \nabla \theta = 0 \\ u = \mathcal{R}^\perp \theta \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (1.29)$$

The origins of this equation lie in the field of geophysical fluid dynamics, see Pedlosky [61] for an introduction from this perspective. One of the main arguments of this thesis is that from all known geometric quantities, SQG (1.29) forms a good model for the full 3D Euler equations. See section 3 for more details.

## 1.2 Infinite Dimensional Geometry

We now change gears. The central purpose of this thesis is to investigate the combination of the previously discussed equations with the infinite dimensional geometric framework which will be introduced in this section.

### 1.2.1 Vector Spaces

A finite dimensional manifold is a space that looks like  $\mathbb{R}^n$  when you zoom in on it close enough. Similarly, an infinite dimensional manifold is an abstract space that looks like an infinite dimensional vector space when you zoom in on it. See Figure 1.1.

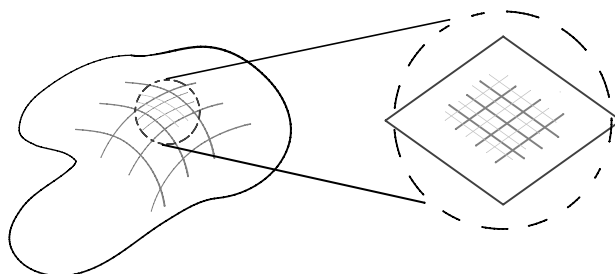


Figure 1.1: Whether finite or infinite dimensional, a manifold is a shape that looks like a vector space when you zoom in close.

We'll refer to this space as the **local modeling space** of our manifold. Every tangent space to the manifold is linearly isomorphic to the local modeling space, so the tangent space critically controls the topological and geometric properties of the underlying manifold. In finite dimensions, every vector space is isomorphic to  $\mathbb{R}^n$ , so the choice doesn't matter so much (unless one does complex geometry). In infinite dimensions there's a wide variety of non-isomorphic vector spaces with distinct properties. The choice of which space to use depends on: convenience, the specific geometries and equations involved, as well as which tools one wishes to have at their disposal. In this thesis we will focus on manifolds of mappings, whose tangent spaces will essentially look like spaces of functions of varying regularities. Let  $U \subset \mathbb{R}^n$  be open with smooth boundary and compact closure. Some of the vector spaces we will be working with in this thesis are given below. See Evans [28] for an introduction to the general theory of these spaces as well as their applicability to the theory of PDE.

- $C^k$  spaces:

$$* C^k(U) = \{u : U \rightarrow \mathbb{R} \mid u \text{ is } k \text{ times differentiable}\}$$

- \*  $C^k(\bar{U}) = \{u : \bar{U} \rightarrow \mathbb{R} \mid D^\alpha u \text{ is uniformly continuous on bounded subsets of } U, \text{ for all } |\alpha| \leq k\}$  with norm

$$\|u\|_{C^k(\bar{U})} = \sum_{|\alpha| \leq k} \left\{ \sup_{x \in U} |D^\alpha u(x)| \right\}$$

- \*  $C^\infty(U) = \bigcap_{k=0}^{\infty} C^k(U)$
- \*  $C^\infty(\bar{U}) = \bigcap_{k=0}^{\infty} C^k(\bar{U})$

- Hölder spaces:

$$* C^{k,\alpha}(\bar{U}) = \left\{ u \in C^k(\bar{U}) \mid \|u\|_{C^{k,\alpha}(\bar{U})} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\alpha}(\bar{U})} \right) < \infty \right\}$$

where

$$[D^\alpha u]_{C^{0,\alpha}(\bar{U})} = \sup_{x,y \in U, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\alpha} \right\}$$

Note that  $\|u\|_{C^{k,\alpha}(\bar{U})}$  is also a norm.

- Sobolev spaces:

- \*  $W^{k,p}(U) = \{u : U \rightarrow \mathbb{R} \mid u \in L^1_{loc}(U) \text{ and for each multiindex } \alpha \text{ with } |\alpha| \leq k, D^\alpha u \text{ exists in the weak sense and belongs to } L^p(U)\}$

with norm

$$\|u\|_{W^{k,p}} = \begin{cases} \left( \sum_{|\alpha| \leq k} \int_U |D^\alpha u|^p dx \right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha| \leq k} \text{ess sup}_U |D^\alpha u| & (p = \infty) \end{cases}$$

- \*  $H^k(U) = W^{k,2}(U)$

- \*  $\dot{H}^k(U) = \{u : U \rightarrow \mathbb{R} \mid u \in L^1_{loc}(U) \text{ and for each multiindex } \alpha \text{ with } |\alpha| = k, D^\alpha u \text{ exists in the weak sense and belongs to } L^2(U)\}$

These definitions can readily be ported to manifolds. In particular, given a (finite dimensional) Riemannian manifold  $(M, g)$ , one can naturally associate a volume form  $\mu$  as is discussed in Lee

[44]. This leads readily to a notion of  $L^p(M)$ . As for the differential conditions, one treats  $U$  as the image of an open set in  $M$  under a chart. In the case that  $M$  is compact this works without much issue. See Marsden-Ebin-Fischer [49] for more on this. In the case that  $M$  is not compact more issues arise, however in this thesis we're principally concerned with the compact case and the non-compact case will be dealt with explicitly when it arises. Note that  $C^\infty(U)$  is a Fréchet space, the rest are Banach and Hilbert spaces.

### 1.2.2 Functions and Differentiability

Let  $X, Y$  be Banach spaces, either  $\mathbb{R}^n$  or one of the Banach spaces listed in the previous section. Let  $U \subset X, V \subset Y$  be open subsets, and  $f : U \rightarrow V$ . Let  $u \in U$ . We say that  $f$  is Gâteaux differentiable if

$$df_u(a) = \lim_{h \rightarrow 0} \frac{f(u + ha) - f(u)}{h} \quad (1.30)$$

exists for all  $a \in X$ . One can immediately see that this is a direct generalization of the usual directional derivative in  $\mathbb{R}^n$ . Unfortunately this notion of differentiability is not enough on its own to recreate theorems such as the inverse function theorem. In particular such a theorem requires conditions such as the derivative of our function being a bounded linear operator that depends continuously on the base-point  $u$ . We need a stronger notion, which will be that of the Fréchet derivative. We say that  $f$  is Fréchet differentiable at  $u \in U$  if there exists a bounded linear operator  $df_u : X \rightarrow Y$  such that

$$\lim_{a \rightarrow 0} \frac{\|f(u + a) - f(u) - df_u(a)\|_Y}{\|a\|_X} = 0. \quad (1.31)$$

In this thesis when we speak of differentiability we will always mean Fréchet differentiability unless otherwise indicated. We may also think of  $df$  as a map:

$$df : U \rightarrow L(X, Y), \quad (1.32)$$

where  $L(X, Y)$  is the collection of bounded linear maps from  $X$  to  $Y$ . This will become handy later on, especially in section 3.3, when we need higher order derivatives,  $d^k f$  which can be thought of as

maps:

$$d^k f : U \rightarrow L(X, L(X, \dots, L(X, Y) \dots)) = L^k(X, Y), \quad (1.33)$$

where  $L^k(X, Y)$  is the space of multilinear maps as in [43]. Now, the following facts are good exercises to think about: if  $f$  is Fréchet differentiable, then it will also be Gâteaux differentiable, but the converse is not always true. Additionally, if you can show that a function is Gâteaux differentiable, with bounded derivatives that depend continuously on the basepoint  $u$ , then it is Fréchet differentiable. This will be our approach in section 3.3 to showing that the Riemannian exponential map associated to the SQG equation is smooth.

### 1.2.3 Manifolds and Manifolds of Mappings

From now on we'll use the notation  $M$  when the manifold in question is finite dimensional, and we'll use  $\mathbf{M}$  when the manifold is infinite dimensional. As in Lang [43] we'll say that a set  $\mathbf{M}$  is a (possibly infinite dimensional)  $C^k$  manifold if it admits an equivalence class of atlases of class  $C^k$ . In general, we will only consider manifolds that are  $C^\infty$  and modeled on a single Banach space  $X$ . The infinite dimensional manifolds that will be the most interesting to us in this thesis will be manifolds of mappings. Let  $M$  and  $N$  be compact manifolds. In this case our underlying sets  $\mathbf{M}$  will be the following possible spaces of maps:

- $C^k(M, N) = \{\eta : M \rightarrow N \mid \eta \text{ is } k \text{ times differentiable.}\}$
- $H^s(M, N) = \{\eta : M \rightarrow N \mid \eta \text{ is in } H^s(\phi(U), \phi(V)) \text{ for every chart } \phi.\}$
- $\mathcal{D}(M) = \{\eta : M \rightarrow M \mid \eta \text{ is a smooth diffeomorphism.}\}$
- $\mathcal{D}^s(M) = \{\eta : M \rightarrow M \mid \eta \in H^s(M, M) \text{ and } \eta^{-1} \in H^s(M, M)\}$
- $\mathcal{D}_\mu(M) = \{\eta \in \mathcal{D}(M) \mid \eta^* \mu = \mu \text{ where } \mu \text{ is the Riemannian volume form of } M\}$

Note:  $H^s(M, N)$  only makes sense when  $s > n/2$  where  $n$  is the dimension of  $M$  and  $\mathcal{D}^s(M)$  only makes sense when  $s > n/2 + 1$ , as is discussed in Marsden-Ebin-Fischer [49]. One will note that

these indices depend entirely on the Sobolev embedding theorem, i.e., we need the maps  $\eta$  to at least be continuous.

The fact that these spaces are infinite dimensional manifolds is largely due to Eells [21] and Ebin-Marsden [18]. See also Marsden-Ebin-Fischer [49] for an accessible introduction. In the case that we wish to study  $\mathbb{R}^n$  we can consider similar spaces, however typically we'll need some sort of boundedness or decay condition. This is due to the fact that the diffeomorphism group of  $\mathbb{R}^n$  is not locally contractible, see Kriegl-Michor [42]. One of the main cases that will arise in this thesis is that of  $C^{k,\alpha}$  class diffeomorphisms of  $\mathbb{R}^n$  in section 3.3 in which case we may think of the space as an affine space. The Sobolev case is done in detail in Inci-Kappeler-Topalov [36]. For completeness we outline the main ideas for compact  $M$  and  $\mathcal{D}(M)$  here.

The main idea is to Figure out what the tangent spaces to  $\mathcal{D}(M)$  should look like. Then we'll be able to use the tangent spaces as our local modeling spaces. From finite-dimensional differential geometry we know that it's quite profitable to look at the derivatives of curves as tangent vectors. So, let

$$\begin{aligned}\eta &: (-1, 1) \rightarrow \mathcal{D}(M) \\ \eta &: t \mapsto \eta(t),\end{aligned}\tag{1.34}$$

such that  $\eta(0) = \eta_0 \in \mathcal{D}(M)$ . What should the derivative of  $\eta$  look like? The key is in the property known as Cartesian closedness. We will take this for granted here, but it's a nontrivial property, again see Kriegl-Michor [42]. The use of the property works as follows. We begin by re-writing  $\eta$  as a map:

$$\begin{aligned}\eta &: (-1, 1) \rightarrow (M \times M) \\ \eta &: t \mapsto (x, \eta(t, x)),\end{aligned}\tag{1.35}$$

and we will instead now think of  $\eta$  as a map:

$$\begin{aligned}\eta &: (-1, 1) \times M \rightarrow M. \\ \eta &: (t, x) \mapsto \eta(t, x).\end{aligned}\tag{1.36}$$

This is perhaps the single most important tool in computing with manifolds of mappings. Notice how we've taken the original map  $\eta$  (1.34), which landed in an abstract space  $\mathcal{D}(M)$  which is difficult

to think about, and turned it into the new map  $\eta$  (1.36) which is a map between finite dimensional manifolds that we can compute anything we'd like with. Fixing an  $x = x_0$  in the domain of  $\eta$  (1.36) we obtain a map:

$$\eta_{x_0} : (-1, 1) \rightarrow M,$$

which is just a curve in our manifold  $M$ . Hence

$$\left. \frac{d}{dt} \right|_{t=0} \eta_{x_0}(t) \in T_{\eta_0(x_0)}M,$$

as in our usual finite-dimensional geometry. Letting  $x$  vary we can now see that

$$\left. \frac{d}{dt} \right|_{t=0} \eta(t) \in \{X : M \rightarrow TM | \pi(X) = \eta_0\},$$

where  $\pi$  is the canonical projection of the tangent bundle  $TM \rightarrow M$ . We define this as the derivative of our original curve  $\eta$  (1.34). In particular we can see that we should select:

$$T_{\eta_0}\mathcal{D}(M) = \{X : M \rightarrow TM | \pi(X) = \eta_0\}.$$

The point here then is that if  $\eta_0 = id$ , the identity map of  $M$ , then  $T_{id}\mathcal{D}(M)$  is precisely smooth vector fields on  $M$ . If  $\eta$  is allowed to vary, then  $X$  looks like a smooth vector field  $u$  composed with  $\eta_0$ , i.e.,  $X = u \circ \eta_0$  for a vector field  $u$  and diffeomorphism  $\eta_0$ . This is essentially saying to treat  $X$  like a vector field, except its vectors will be based at  $\eta_0(x)$  rather than  $x$ .

The next issue that we come to is defining charts. We need a way of moving from vector fields to diffeomorphisms. Perhaps the first thing to try is using the flow of each vector field to construct the diffeomorphisms. This map ends up being the Lie group exponential map of  $\mathcal{D}(M)$ . Unfortunately this doesn't work as a chart as this map is not locally surjective, i.e., there are diffeomorphisms that are not the flow of any vector field as is shown in Hamilton [33]. Instead we must employ the Riemannian exponential map of the underlying manifold. Again, see Eells [21] or Marsden-Ebin-Fischer [49] for more information.

In infinite dimensional geometry, a major distinction is drawn between using Banach spaces as local modeling spaces and using Fréchet spaces. Banach spaces have the advantage that they

have an implicit function theorem, so a large amount of finite dimensional manifold theory can be reproduced in a similar fashion with minor modifications as in Lang [43]. That being said, throughout this thesis we will make heavy use of the idea that  $\mathcal{D}(M)$  and  $\mathcal{D}_\mu(M)$  are infinite-dimensional Lie groups, which will be discussed in the next section 1.2.4. There is a result due to Omori [59] which states that a Banach Lie group acting faithfully and effectively on a finite dimensional manifold  $M$  is itself finite dimensional, which means in particular that none of our Banach manifolds of mappings (such as  $\mathcal{D}^s(M)$  and  $\mathcal{D}_\mu^s(M)$ ) can ever be legitimate Lie groups. What happens in practice is that their right translation is smooth, but their left translation isn't Lipschitz continuous as is discussed in Ebin-Marsden [18]. This tradeoff means that we'll use these different regularities of manifolds of mappings for different purposes.

#### 1.2.4 Right Invariant Metrics

The main motivation for this field of study comes from the ideas of Arnold [1], where he observed that  $\mathcal{D}_\mu(M)$  can be thought of as an infinite dimensional Lie group, and once this manifold is endowed with the  $L^2$  Riemannian metric, its geodesics (locally length-minimizing curves) become equivalent to solutions to the Euler equations (1.1). Ebin-Marsden [18] used these ideas to write the Euler equations (1.1) as an ODE on  $\mathcal{D}_\mu^s(M)$  which could then be solved with the contraction mapping principle. More generally, given a Lie group  $G$ , we can endow the group with a right-invariant metric whose geodesics are given by some ODE or PDE. The surprising thing is that many of these equations are important equations from mathematical physics. We call this category of equations Euler-Arnold equations and in fact, one of the main perspectives of this thesis is that the Wunsch equation, (1.27) with  $b = 2$ , and SQG (1.29) are Euler-Arnold equations. See Table 1.1 for a list of many such equations. While this section is an overview of these concepts, Arnold-Khesin [2] has more details.

Let  $G$  be a (possibly infinite-dimensional) Lie group. For the purposes of this thesis, this will mean that  $G$  is a Fréchet manifold and in particular will either be  $\mathcal{D}(M)$  or  $\mathcal{D}_\mu(M)$ , although there are many other possibilities. For this discussion in particular we will mostly consider the example

where  $G = \mathcal{D}(M)$ , the case  $G = \mathcal{D}_\mu(M)$  will be discussed towards the end of the section.  $G$  will also have a group operation,  $*$  :  $G \times G \rightarrow G$  that is Fréchet differentiable, i.e., compatible with the smooth structure of the Fréchet manifold. In the case of  $\mathcal{D}(M)$  and  $\mathcal{D}_\mu(M)$  this will be given by the composition of diffeomorphisms, for example:

$$\circ : \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow \mathcal{D}(M)$$

$$\eta \times \zeta \mapsto \eta \circ \zeta.$$

Fixing a diffeomorphism  $\zeta$ , we obtain the right translation map:

$$R_\zeta : \mathcal{D}(M) \rightarrow \mathcal{D}(M)$$

$$\eta \mapsto \eta \circ \zeta.$$

Similarly we could fix  $\eta$  and obtain left translation  $L_\eta$ . Smoothness of  $\circ$  gives us smoothness of each of these maps. Taking differentials, one can show that push-forward by right translation is given by:

$$d(R_\zeta)_\eta : T_\eta \mathcal{D}(M) \rightarrow T_{\eta \circ \zeta} \mathcal{D}(M)$$

$$U \mapsto U \circ \zeta$$

Now, recall that for a Lie group  $G$ , the Lie algebra is denoted by  $\mathfrak{g} = T_e G$  where  $e$  is the identity element of the group. Hence in the case of  $\mathcal{D}(M)$ ,  $e = id$ , the identity map, and  $\mathfrak{g} = T_e \mathcal{D}(M) = \chi(M)$ , the collection of smooth vector fields on  $M$ , which follows immediately from the discussion in the previous section, (1.2.3). Now, given an inner-product on  $T_e G$ , we can extend the inner-product to the other tangent spaces  $T_g G$  for other  $g \in G$  using push-forward by either left or right translation. In the case of  $\mathcal{D}(M)$  we denote the inner-product  $\langle\langle \cdot, \cdot \rangle\rangle_e$  to distinguish it from the inner-product  $\langle \cdot, \cdot \rangle$  on the underlying Riemannian manifold  $(M, g)$ . For example, we could take the  $L^2$  inner product for  $u, v$ :

$$\langle\langle u, v \rangle\rangle_e^{L^2} = \int_M \langle u, v \rangle d\mu. \quad (1.37)$$

To obtain the metric at other tangent spaces  $T_\eta \mathcal{D}(M)$ , recall that all vectors  $U, V \in T_\eta \mathcal{D}(M)$  are of the form  $U = u \circ \eta, V = v \circ \eta$ . Hence the full right invariant  $L^2$  metric is given by

$$\langle\langle U, V \rangle\rangle_\eta^{L^2} = \langle\langle U \circ \eta^{-1}, V \circ \eta^{-1} \rangle\rangle_e^{L^2} = \int_M \langle u, v \rangle d\mu. \quad (1.38)$$

One of the main strengths of the Lie group approach to fluid mechanics and geometry in general is that when we're on a Lie group with a right-invariant metric, we can exploit right-translation to write down essentially all Riemannian geometric quantities on the Lie algebra,  $\mathfrak{g}$ . This includes the geodesic equation, curvature equations and Jacobi equation (see section (1.2.5)). This process is equivalent to the concept of the Eulerian vs. Lagrangian points of view in classical mechanics. We will begin by writing down the general form of the geodesic equation for a right invariant metric on the Lie algebra. Any equation of this form is an **Euler-Arnold equation**. If we weren't on a Lie group with a right invariant metric we would have to write down the connection via the Koszul formula to obtain:

$$\nabla_{\eta_t} \eta_t = 0 \tag{1.39}$$

We would also have to be careful about things like uniqueness of the connection. In fact, if we were working with a non-invariant metric (see the example below) then this is what we'd have to do. See Constantin-Kolev [13] for much more detail on this approach. Instead, we will, as said above, write this down as an equation purely in terms of the Lie algebra as in Arnold's approach: [1] and [2]. First we let

$$\text{ad} : T_e \mathcal{D}(M) \times T_e \mathcal{D}(M) \rightarrow T_e \mathcal{D}(M) \tag{1.40}$$

$$u, v \mapsto \text{ad}_u v = -[u, v],$$

where  $[u, v]$  is the usual Lie bracket of vector fields.  $\text{ad}$  is a concept that extends to general Lie groups, again see Arnold-Khesin [2] for more details. The geodesic equation on  $\mathcal{D}(M)$  can then be expressed as

$$\begin{cases} u_t = -\text{ad}_u^* u & (1.41) \\ \eta_t = u(t, \eta), & (1.42) \end{cases}$$

where we take  $\text{ad}^*$  to be the adjoint of  $\text{ad}$  with respect to  $\langle \langle \cdot, \cdot \rangle \rangle_e$ . I.e.,

$$\langle \langle \text{ad}_u^* v, w \rangle \rangle_e = \langle \langle v, \text{ad}_u w \rangle \rangle_e \tag{1.43}$$

for all  $u, v, w \in \mathfrak{g}$ .

**Remark.** *This choice for the notation  $\text{ad}^*$  is consistent with Misiołek-Preston [53], which is different than the notation in Arnold-Khesin [2].*

The first equation above, (1.41), is known as the  $\text{ad}^*$  equation which we call the Euler-Arnold equation, and the second equation, (1.42), is known as the flow equation. The idea is that we solve (1.41) for  $u$  on  $T_eG$ , and then solve (1.42) for  $\eta$  which then ports the solution to the Lie group itself. When solved together, (1.41) and (1.42) yield the same solution as (1.39). We will see later on other quantities, such as the Jacobi equation that split in a similar way when ported to the Lie algebra.

**Example:** We now demonstrate that the right-invariant inviscid Burgers equation is the geodesic equation of  $\mathcal{D}(S^1)$  in the right-invariant  $L^2$  metric (1.38). To do so we must compute  $\text{ad}^*$  and plug it into equation (1.40). Let  $u, v \in T_e\mathcal{D}(S^1)$ . Going straight from the definition:

$$\begin{aligned} \langle \langle \text{ad}_u^* u, v \rangle \rangle_e &= \langle \langle u, \text{ad}_u v \rangle \rangle_e \\ &= \int_{S^1} u \text{ad}_u v dx = \int_{S^1} u(-[u, v]) dx \\ &= \int_{S^1} u(u_x v - uv_x) dx = \int_{S^1} 3uu_x v dx \end{aligned}$$

Since  $v$  was arbitrary, this implies that

$$\text{ad}_u^* u = 3uu_x$$

Hence equation 1.41 becomes

$$u_t + 3uu_x = 0. \tag{1.44}$$

It's important to note that this equation is fundamentally different from the more well known non-invariant burgers equation:

$$u_t + uu_x = 0, \tag{1.45}$$

which arises when considering the non-invariant  $L^2$  metric on  $\mathcal{D}(S^1)$  given by, for  $U, V \in T_\eta\mathcal{D}(M)$  such that  $U = u \circ \eta$  and  $V = v \circ \eta$  for  $u, v \in T_{id}\mathcal{D}(M)$ :

$$\langle\langle U, V \rangle\rangle = \int_{S^1} UV dx = \int_{S^1} \frac{uv}{\eta_x} dx. \quad (1.46)$$

This should be contrasted with the right-invariant  $L^2$  metric (1.38). Intuitively, if we think of  $u$  as the velocity field of a fluid flow  $\eta$ , in the invariant metric we only measure how large the velocity field is to find the energy of the flow. In the non-invariant metric we weight not only the velocity field but also the positions of the particles. For example, if our fluid is compressed then  $\eta_x$  will be small causing (1.46) to be large. The non-invariant  $L^2$  metric corresponds much more closely with physical intuition. As for the equations (1.44) and (1.46), at the Eulerian level, one is a simple scaling of the other. However, non-invariant Burgers (1.46) ends up being much simpler at the Lagrangian level. In particular its characteristics coincide with its geodesics. This is **not** true for right-invariant Burgers (1.44). Its characteristics remain fairly simple, but its geodesics become much more complicated. See Constantin-Kolev [13] and Disconzi-Ebin-Misiołek-Preston [16] for more in depth discussions of Burgers equation and these issues. Given this argument of the non-physical nature of right-invariant metrics, it seems remarkable that so many important equations from mathematical physics appear as their geodesic equations as in Table 1.1 below.

**Warning.** *This has been an informal discussion which obscures certain technical difficulties, the most important of which is the observation that the  $L^2$  metric makes a topology on  $T_{id}\mathcal{D}(M)$  that is **incompatible** with the topology on  $\mathcal{D}(M)$ . Such a metric is called a **Weak Riemannian Metric**. This issue is one of the central sources of difficulties in this field. In particular, in finite dimensional Riemannian geometry, if one is given a Riemannian manifold, then one is guaranteed of many facts, like the fact that the metric and Riemannian exponential map (see below) are smooth. These things frequently need to be re-proven from scratch in the Weak Riemannian case. For example, in section 3.3 We demonstrate this for the Riemannian exponential map for the  $\dot{H}^{1/2}$  metric on the space of Hölder and Sobolev class diffeomorphisms of  $\mathbb{R}$ .*

Manifold	Metric	Geo. Equation	Global Existence	Curvature	Fredholm exp
1D $M$ :					
$\mathcal{D}(S^1)$	$L^2$	R-I Burgers [13]	No [13]	Positive [38]	N/A [16]
$\mathcal{D}(S^1)$	$\dot{H}^{1/2}$	Wunsch [83]	No [8], [4], [67]	Unknown [4], [7]	No [4]
$\mathcal{D}(S^1)$	$\dot{H}^1$	Hunter-Saxton [37]	No [35]	Constant [46]	Yes [39]
$\mathcal{D}(S^1)$	$\dot{H}^{3/2}$	EWP [31]	Yes [31]	Negative [80]	Likely
$\mathcal{D}(S^1) \times \mathbb{R}$	$L^2$	KdV [60]	Yes [73]	Both signs [52]	N/A [14]
2D $M$ :					
$\mathcal{D}_\mu(M)$	$L^2$	Inc. Euler [1], [18]	Yes [85], [18]	Both signs [1]	Yes [20]
$\mathcal{D}_\mu(M)$	$\dot{H}^{1/2}$	SQG [76], [81]	Unknown	Both signs [81]	No [81]
3D $M$ :					
$\mathcal{D}_\mu(M)$	$L^2$	Inc. Euler [1], [18]	Unknown	Both signs [1]	No [20]

Table 1.1: A collection of diffeomorphism groups with right-invariant metrics and their associated Euler-Arnold equations

### 1.2.5 The Jacobi Equation and Conjugate Points

Let  $\mathbf{M}$  be a possibly infinite dimensional Riemannian manifold with Riemannian metric  $\langle\langle \cdot, \cdot \rangle\rangle$ .

The **Riemannian exponential map** is given as follows:

**Definition 1.2.1.** *Let  $p \in \mathbf{M}$ . We define*

$$\begin{aligned} \exp_p : T_p\mathbf{M} &\rightarrow \mathbf{M} \\ u_0 &\rightarrow \eta(1) \end{aligned} \tag{1.47}$$

where  $\eta$  is the geodesic satisfying  $\eta(0) = p$  and  $\eta_t(0) = u_0$ .

As the geodesics of  $\mathbf{M}$  are critically linked to its geometry, it's natural to mount a study of the Riemannian exponential map. First note that we may express any geodesic satisfying  $\eta(0) = p$ ,  $\eta(1) = q$ , and  $\eta'(0) = u_0$  as  $\eta(t) = \exp_p(tu_0)$ . Let  $\alpha(s) = u_0 + sv_0$  be a curve in  $T_p\mathbf{M}$ . We may

then think of  $d(\exp_p)_u$  as a map

$$d(\exp_p) : T_{u_0}(T_p\mathbf{M}) \rightarrow T_q\mathbf{M}$$

$$v_0 \mapsto \left. \frac{d}{ds} \right|_{s=0} \exp_p(u_0 + sv_0).$$

What should all this represent geometrically? Well, the collection of curves  $\Gamma(s, t) = \exp_p(t(u_0 + sv_0))$  will be a family of geodesics in  $\mathbf{M}$ . The derivatives of these curves in  $s$  are vector fields known as Jacobi fields. More precisely, these are the Jacobi fields vanishing at the identity. As in Lee ([44], pg. 174), suppose that we have a family of curves  $\Gamma : (-\epsilon, \epsilon) \times [a, b] \rightarrow M$ . Let  $J : (-\epsilon, \epsilon) \times [a, b] \rightarrow M$  be a vector field along  $\Gamma$  known as a variation field.  $\Gamma$  is said to be a variation through geodesics if  $\Gamma(s_0, t)$  is a geodesic for every  $s_0 \in (-\epsilon, \epsilon)$ . The following theorem from Lee [44] characterizes variation fields through geodesics:

**Theorem 7** (Lee [44]). *Let  $\eta$  be a geodesic and  $J$  a vector field along  $\eta$ . If  $J$  is the variation field of a variation through geodesics then  $J$  satisfies the **Jacobi Equation**:*

$$D_t^2 J + R(J, \eta_t)\eta_t = 0. \tag{1.48}$$

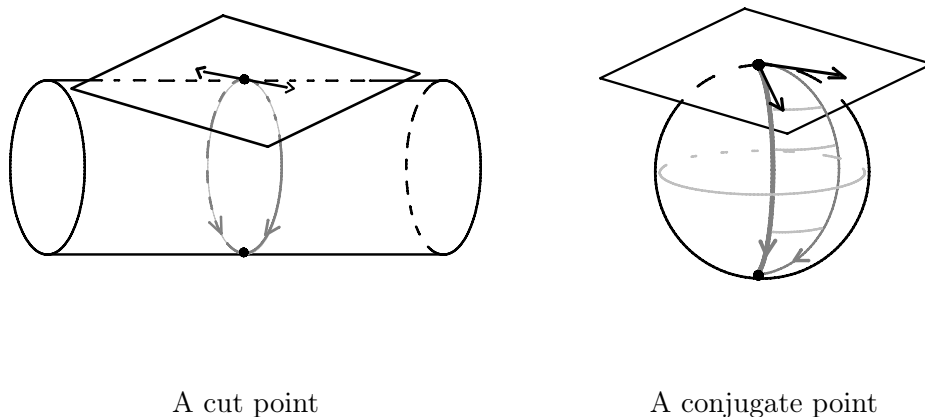
The key point of all of this is that the Jacobi equation gives us a concrete way to compute the differential of  $\exp_p$ . That is, at a time  $t$  a solution to the Jacobi equation  $J$  satisfying  $J(0) = 0$  is a vector  $J(t) = d(\exp_p)_{u_0}(v_0)$ . Note that Lee [44] only deals with finite dimensional manifolds, however these results carry over to diffeomorphism groups. A good reference for the corresponding discussion of this section is Misiołek-Preston [53]

The critical points of  $\exp_p$  will be vectors  $u_0 \in TT_p\mathbf{M} \cong T_p\mathbf{M}$  for which  $\exp_p(t(u_0 + sv_0))$  doesn't change with respect to  $s$  and  $v_0$  up to first order. These vectors are of great importance in this dissertation. We have the following:

**Definition 1.2.2.** *The critical values of  $\exp_p$  are known as the **conjugate points** of  $\mathbf{M}$  with respect to  $p$ .*

Another way of phrasing all this is that if  $q$  is a conjugate point of  $p$ , then the geodesics  $\eta_1$  and  $\eta_2$  satisfying  $\eta_1(0) = p$ ,  $\eta_2(0) = p$ ,  $\eta_1'(0) = u_0$ , and  $\eta_2'(0) = u_0 + sv_0$  for  $s$  small will meet up at

$q$  up to first order. Note that this is no guarantee that they do actually meet: this is the content of the Morse-Littauer theorem [55] for finite dimensional manifolds, which says that given a conjugate point, two geodesics will intersect at that point. For constant curvature 2D manifolds the behavior of a family of geodesics strongly depends on the underlying curvature. For flat Euclidean space, geodesics will diverge linearly in time. For negatively curved hyperbolic space geodesics will spread apart rapidly. Finally, for the positively curved sphere, geodesics will start at  $p$  and converge and meet at the point on the sphere antipodal to  $p$ . When two geodesics on  $\mathbf{M}$  leaving a point  $p$  collide at a point  $q$ , we say that  $q$  is a **cut point**. Another way to think of  $\exp_p$  is as an embedding of a neighborhood  $U \subset T_p\mathbf{M}$  into  $\mathbf{M}$ . When we have a cut point  $q$ ,  $\exp_p$  fails to be an embedding since it's no longer injective. When we have a conjugate point,  $\exp_p$  fails to be an immersion, since  $q$  corresponds to a critical point. See Figure 1.2.



A cut point

A conjugate point

Figure 1.2: cut points vs. conjugate points

Another way to think of a cut point is it occurs when two geodesics happen to collide. A conjugate point occurs when a whole infinite family of geodesics collides. In infinite dimensions there is an additional complication in that when we think of as a linear map  $d\exp : T_u(T_p\mathbf{M}) \cong T_p\mathbf{M} \rightarrow T_q\mathbf{M}$ , it can fail to be invertible in a variety of ways. In particular, when  $d\exp$  fails to be invertible but is a Fredholm map, we say that  $q$  is **monoconjugate** to  $p$ . It can also happen that  $d\exp$  not only fails to be invertible but also fails to be Fredholm. In this case we call  $q$  an **epiconjugate**

point to  $p$ . It was shown in Ebin-Misiołek-Preston [20] that the 2D volumorphism group,  $\mathcal{D}_\mu(M)$  with the  $L^2$  metric is Fredholm, i.e., its Riemannian exponential map is a nonlinear Fredholm in the sense of Smale [74], while the 3D volumorphism group is **not** Fredholm. This points to a crucial difference between 2D and 3D fluid mechanics. In a sense, the geometry of the volumorphism group of a 2D manifold behaves in many ways like a finite dimensional manifold, while the geometry of a 3D volumorphism group is truly infinite dimensional.

When  $\mathbf{M} = \mathcal{D}_\mu(M)$  for  $M$  a 2D manifold, Misiołek [54] showed that  $\mathbf{M}$  satisfies the Morse-Littauer theorem. This result critically relies on the fact that in this case,  $\exp$  is a nonlinear Fredholm map. In other words,  $d\exp$  is a Fredholm map. This Fredholm property seems to be critically connected not only with other geometric properties such as the curvature and an ability to replicate aspects of Morse theory, but also with blow up of the geodesic equation. See Table 1.1 for a partial list of groups, metrics, and their Fredholmness properties.

Now, recall that in the previous section we took the geodesic equation (1.39) on  $\mathcal{D}(M)$  and discussed how it could be ported to the Lie algebra  $\mathfrak{g}$  via the  $\text{ad}^*$  formula (1.41) and the flow equation (1.42). It was shown in Rouchon [69] and Preston [62] that one can do the same to the Jacobi equation and that one obtains a similar splitting. The result is the following proposition taken from Misiołek-Preston [53]:

**Proposition 1.2.1** (Misiołek-Preston [53]). *Suppose  $G$  is any Lie group with a (possibly weak) right-invariant metric. Let  $\eta(t)$  be a smooth geodesic with  $\eta(0) = e$  and  $\dot{\eta}(0) = u_0$ . Then, every proper Jacobi field  $J(t)$  (such that  $J(0) = 0$ ) along  $\eta$  satisfies the following system of equations on  $T_eG$ :*

$$\frac{dY}{dt} - \text{ad}_u Y = Z \tag{1.49}$$

$$\frac{dZ}{dt} + \text{ad}_u^* Z + \text{ad}_Z^* u = 0, \tag{1.50}$$

where  $J(t) = dR_{\eta(t)}Y(t)$ ,  $\dot{\eta}(t) = dR_{\eta(t)}u(t)$ ,  $Y(0) = 0$ , and  $Z(0) = 0$ .

The first equation (1.49) is known as the linearized flow equation and the second equation (1.50) is the linearized Euler equation. We'll make heavy use of these formulas in sections 3.4 and 4.3.

Finally, on a Riemannian manifold, the **Morse index form** is given by:

$$I(J, J) = \int_a^b \left\| \frac{DJ}{dt} \right\|^2 - \langle \langle R(J, \dot{\eta})\dot{\eta}, J \rangle \rangle dt, \quad (1.51)$$

where  $J$  is a Jacobi field along a given geodesic  $\eta$ . In a way similar to the Euler-Arnold equation 1.41 and 1.42, as well as the Jacobi equation 1.48, on a Lie group the Morse index form can be written on the Lie algebra  $\mathfrak{g}$ . The only difference is that this time using pushforward by left translation yields a very useful formula. This and the broader significance of the Morse index form are captured in the following lemma from Bauer-Kolev-Preston [4]:

**Lemma 1.2.1** (Bauer-Kolev-Preston [4]). *Suppose  $G$  is a Lie group with a weak right-invariant metric  $\langle \langle \cdot, \cdot \rangle \rangle$ , and let  $\eta$  be a geodesic in  $G$  with  $\eta(0) = id$  and  $\dot{\eta}(0) = u_0$ , defined on  $[0, T]$ . Then for any  $a, b$  with  $0 \leq a < b \leq T$ , the Morse index form for a Jacobi field  $J = dL_\eta v$  is given by*

$$I(J, J) = \int_a^b \left\| Ad_{\eta(t)} \dot{v}(t) \right\|^2 + \langle \langle u_0, ad_{v(t)} \dot{v}(t) \rangle \rangle dt. \quad (1.52)$$

*If the index form is negative for some field  $v$  with  $v(a) = v(b) = 0$ , then  $\eta(a)$  is monoconjugate to  $\eta(b - \epsilon)$  for some  $\epsilon > 0$ , in the sense that there is a Jacobi field with  $J(a) = J(b - \epsilon) = 0$ .*

What's really valuable here is that the left translated Morse index formula can be computed in terms of quantities that are readily available on diffeomorphism groups. See Misiołek-Preston [53] for computations of  $Ad$ ,  $ad$ , etc. on a variety of such groups.

### 1.3 The Setting for the $\dot{H}^{1/2}$ metric

As has been indicated before, the primary focus of this dissertation will be the study of the right-invariant  $\dot{H}^{1/2}$  metric on either  $\mathcal{D}(S^1)$  or the  $\dot{H}^{-1/2}$  metric on  $\mathcal{D}_\mu(M)$  where  $M$  is a 2D manifold (which turns into the  $\dot{H}^{1/2}$  metric on stream functions. In the first case, the Euler-Arnold equation is the Wunsch equation, which was discovered by Wunsch [83]. In the second case, it's the SQG equation, which was essentially found by Tao [76], but put into an Euler-Arnold framework in Washabaugh [81] as well as in this dissertation (see Chapter 4). There has been a significant

geometric theory already built up for the Wunsch equation, which can be found in Bauer-Kolev-Preston [4]. In this section we discuss some of the remarkable properties of this space that were discovered in this paper.

In Preston-Washabaugh [67] it was recognized that the  $\dot{H}^{1/2}$  metric has in fact already been investigated by Teo [77] from the perspective of Teichmüller theory. Preston-Washabaugh [67] is reproduced and expanded upon in Chapter 3, in it we begin the investigation of this correspondence from a Euler-Arnold perspective. Hence in this section we'll also lay out some of the connections to Teichmüller theory as preparation.

### 1.3.1 The right-invariant $\dot{H}^{1/2}$ metric on $\mathcal{D}(S^1)$

In Bauer-Kolev-Preston, the authors found several geometric properties of the right-invariant  $\dot{H}^{1/2}$  metric on  $\mathcal{D}(S^1)$ :

- They found that the Riemannian exponential map is non-Fredholm. They also established a criteria for the existence of conjugate points along a geodesic.
- They extended a criteria from Castro-Córdoba [8] for solutions to the Wunsch equation to fail to exist in finite time.
- They found that the sectional curvature can be come arbitrarily large in magnitude. The authors conjectured that the curvature is positive, and while computations were done in Chhay [7] that further suggest this, the question remains open.

In this section, it is the first and second of these properties that we wish to discuss in more detail.

#### 1.3.1.1 Conjugate points on $\mathcal{D}(S^1)$ in the right-invariant $\dot{H}^{1/2}$ metric

In Bauer-Kolev-Preston [4] the authors found the following criteria for the existence of conjugate points along arbitrary geodesics by analyzing the Morse Index formula 1.52:

**Theorem 8** (Bauer-Kolev-Preston [4]). *Let  $\eta$  be a smooth geodesic in  $\text{Diff}(S^1)$  in the  $\mu H^{1/2}$  metric which is defined on the time interval  $[0, T]$ . Let  $0 < a < b < T$ . Then there is some constant  $R$  such that  $\eta$  is not minimizing on  $[a, b]$  whenever, for some  $x_0 \in S^1$ , we have the inequality*

$$|\omega_0(x_0)| \int_a^b \frac{d\tau}{\eta_x(\tau, x_0)^2} > R\pi. \quad (1.53)$$

*For example,  $R = 4/3$  works.*

In section 2.4 we will begin the investigation of the possibility of the existence of a **conjugate cascade**, an infinite sequence of points conjugate to one-another, as was first discussed in Preston [65]. The idea here is that it is possible that from a geometric perspective, blow-up of the geodesic equation corresponds to such a collection of points. As is discussed in section 2.4, there is good reason to believe such a collection exists. We will start the analysis of this problem, but unfortunately the question remains open to this day.

### 1.3.1.2 The Ermakov-Pinney Equation and Finite Time Singularities

An especially beautiful picture of finite time singularities from the Lagrangian perspective (i.e. in terms of the geodesics on  $\mathcal{D}(S^1)$ ) was found in Bauer-Kolev-Preston. The **Ermakov-Pinney equation** is known in classical mechanics as the equation describing the motion of a particle in a central force field as is described in Eliezer-Gray [23].

**Definition 1.3.1.** *The Ermakov-Pinney equation is given by:*

$$r''(t) + \Omega(t)^2 r(t) = \frac{c^2}{r(t)^3}$$

*where  $r(t)$  is the radial distance of the particle from the origin,  $\Omega(t)^2$  corresponds to the central force field, and  $c$  is the constant angular momentum.*

Bauer-Kolev-Preston proved the following theorem:

**Theorem 9** (Bauer-Kolev-Preston [4]). *Suppose  $u$  and  $\omega$  form a solution of the Wunsch equation (equation (1.27) with  $b=2$ ) with  $\int_0^{2\pi} u_0(\theta)d\theta = 0$ . Let  $\eta$  denote the Lagrangian flow of  $u$  satisfying*

$$\eta_t(t, \theta) = u(t, \eta(t, \theta)), \eta(0, \theta) = \theta.$$

*Then  $\eta_x$  satisfies*

$$\eta_{\theta tt}(t, \theta) = \frac{\omega_0(\theta)^2}{\eta_\theta(t, \theta)^3} - F(t, \eta(t, \theta))\eta_\theta(t, \theta),$$

*where  $F(t, \theta) = -uu'' - H(uHu'')$  is positive for all  $t$  and  $\theta$ .*

In other words, from the Lagrangian perspective we may think of the solutions to the Wunsch equation as a collection of particles orbiting with distance  $\eta_\theta$  from the origin and angular momentum of  $\omega_0(\theta)$ , and central forcing term given by the function  $F$ . They had demonstrated the last statement of the theorem in a previous corollary and it is important enough to write down again, that is for any function  $u : S^1 \rightarrow \mathbb{R}$  for which its fourier series converges to itself:

$$F = -uu'' - H(uHu'') > 0 \tag{1.54}$$

This now gives an elegant picture for the formation of singularities in the Wunsch equation. If the set

$$\{\theta_0 : \omega_0(\theta_0) = 0 \text{ and } u'_0(\theta_0) < 0\} \tag{1.55}$$

is non-empty, then in the Ermakov-Pinney representation, such a  $\theta_0$  may be thought of as particle with no momentum, which will dive in towards the origin since the force  $F$  is positive towards the origin. Once  $\eta_x(\theta_0) = 0$ , the geodesic leaves the diffeomorphism group which causes a finite time singularity. In Figure 1.3 we can observe the time evolution of this situation. The shape starts out as a circle and then evolves into the shapes shown with time. The simulation, based off of the simulations done in Preston-Washabaugh [67] clearly shows the points  $\eta_x(\theta_0)$  diving towards the origin. The fascinating part of this is that we may use physical intuition about a particle in a central force field to analyze the behavior of this vorticity model equation. In Preston-Washabaugh [67], section 12 of this thesis, we demonstrate that this blowup criterion, the fact that the set 1.55

is non-empty, in fact applies to **every** mean zero function on the circle and in the case of  $\mathbb{R}$  as well with certain restrictions. Additionally note the similarity of the set (1.55) to the set (1.24). Indeed these results can be ported to the Constantin-Lax-Majda equation itself.

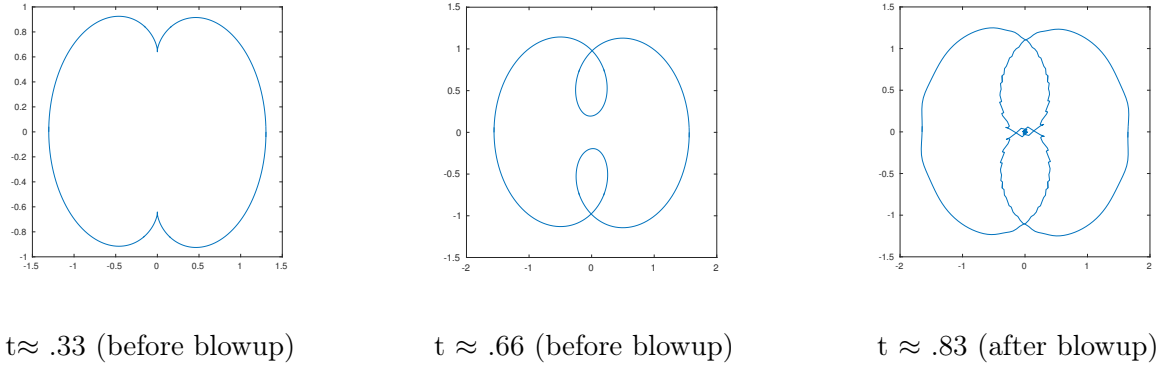


Figure 1.3: Ermakov-Pinney Trajectories for Wunsch with  $u_0 = \sin(2x)$

### 1.3.2 Teichmüller Theory

Teichmüller theory investigates the problem of studying the moduli spaces of genus  $g$  complex curves equivalent up to diffeomorphism. While there are many works such as Nag [57] that study this theory from a complex geometry perspective, Tromba [79] laid down the foundations of the theory from an infinite dimensional geometry perspective. In this context, the  $\dot{H}^{3/2}$  metric is the Weil-Petersson metric, which gained substantial attention in shape space geometry following the work of Feiszli-Mumford [29]. There, the authors employed conformal welding, a process which will be discussed more in Section 2.2. For now one may think of it as an approximate isomorphism between  $\mathcal{D}(S^1)$  and the space of curves in the plane. Feiszli and Mumford used conformal welding and the Weil-Petersson metric to calculate the distances between such curves, a technique that has many applications in fields such as computational anatomy and evolutionary biology.

An Euler-Arnold framework for this metric was investigated by Gay-Balmaz and Ratiu [31]. Amongst other things, they showed that the solutions to the Euler-Weil-Petersson equation (EWP), the corresponding Euler-Arnold equation, are geodesics on a **strong** Riemannian manifold and that

they exist for all time. One of the main results of the next section, Theorem 13, essentially states that solutions to this equation with  $\dot{H}^s$  initial data remain  $\dot{H}^s$  for all time.

One of our main interests in the Teichmüller theory perspective comes from the correspondence between the  $\dot{H}^{1/2}$  metric on  $\mathcal{D}(S^1)/S^1$  and the Velling-Kirillov metric on the universal Teichmüller curve which was discovered in the work of Teo [77]. Teo established this correspondence by using a slightly modified version of conformal welding. In section 2.2, we numerically map the geodesics on  $\mathcal{D}(S^1)/S^1$  in the right-invariant  $\dot{H}^{1/2}$  metric to the corresponding space of curves in the plane. This correspondence has many exciting possibilities which will be discussed in section 2.5.

## Chapter 2

### The Geometry of Vorticity Model Equations

The following paper Preston-Washabaugh [67] was written by both Steve Preston and myself. This version has a few differences from the version submitted for publication. In particular, the sections on Conformal welding, section 2.2, and interpolating  $L^2$  and  $\dot{H}^1$ , section 2.4, have been added in.

#### 2.1 Introduction

Euler-Arnold equations are PDEs that describe the evolution of a velocity field for which the Lagrangian flow is a geodesic in a group of smooth diffeomorphisms of a manifold, for some choice of right-invariant Riemannian metric; see Arnold-Khesin [2]. In the one-dimensional case, we will consider the diffeomorphism group of the circle  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . If the Riemannian metric is defined at the identity by

$$\langle u, u \rangle_r = \int_{S^1} u \Lambda^{2r} u \, d\theta, \quad (2.1)$$

where  $\Lambda^{2r}$  is a symmetric, positive pseudodifferential operator of order  $r$ , we call it a Sobolev  $H^r$  metric, and the Euler-Arnold equation is given by

$$m_t + um_\theta + 2mu_\theta = 0, \quad m = \Lambda^{2r} u, \quad u = u(t, \theta), \quad u(0) = u_0 \in C^\infty(S^1). \quad (2.2)$$

Special cases include the Camassa-Holm equation when  $r = 1$  and  $\Lambda^2 = 1 - \partial_\theta^2$ , or the right-invariant Burgers' equation when  $r = 0$  and  $\Lambda^0 = 1$  [13]. One can also allow  $\Lambda^{2r}$  to be degenerate (nonnegative rather than positive); the best known example is when  $r = 1$  and  $\Lambda^2 = -\partial_\theta^2$ , for which we get the

Hunter-Saxton equation [37]. Here we are interested in the fractional order cases  $r = \frac{1}{2}$  and  $r = \frac{3}{2}$  (see Escher-Kolev [26]), which arise naturally in Teichmüller theory [31]. Both cases are critical in some sense, due to the Sobolev embedding being critical: for  $r < \frac{1}{2}$  Lagrangian trajectories do not depend smoothly on initial conditions, while for  $r > \frac{3}{2}$  conservation of energy is strong enough to ensure global existence [27]. In this paper we will show that all solutions for  $r = \frac{1}{2}$  blow up in finite time while for  $r = \frac{3}{2}$  all smooth solutions exist globally; previously only some solutions were known to blow up in the  $r = \frac{1}{2}$  case [4] and smooth solutions were only known to stay in  $H^{3/2}$  in the  $r = \frac{3}{2}$  case [31].

Specifically the cases we are interested in are:

- ( $r = \frac{1}{2}$ ) the Wunsch equation [83],[4]:  $\Lambda^1 = Hu_\theta$ ,
- and ( $r = \frac{3}{2}$ ) the Euler-Weil-Petersson equation [31]:  $\Lambda^3 = -H(u_{\theta\theta\theta} + u_\theta)$ ,

where  $H$  is the Hilbert transform defined for periodic functions by  $H(e^{in\theta}) = -i \operatorname{sign} n e^{in\theta}$ . The Wunsch equation is a special case of the modified Constantin-Lax-Majda equation [58] which models vorticity growth in an ideal fluid.

When paired with the flow equation

$$\frac{\partial \eta}{\partial t}(t, \theta) = u(t, \eta(t, \theta)), \quad \eta(0, \theta) = \theta, \quad (2.3)$$

the Euler-Arnold equation (2.2) describes geodesics  $\eta(t)$  of the right-invariant Riemannian metric defined at the identity element by (2.1) on the homogeneous space  $\operatorname{Diff}(S^1)/G$ . Here  $G$  is the group generated by the subalgebra  $\ker \Lambda$  of length-zero directions: for the Euler-Weil-Petersson equation we have  $G = \operatorname{PSL}_2(\mathbb{R})$ , and for the Wunsch equation we have  $G = \operatorname{Rot}(S^1) \cong S^1$ .

The local existence result is that if  $u_0 \in H^s(S^1)/\mathfrak{g}$  for  $s > \frac{3}{2}$  (where  $\mathfrak{g}$  is the Lie algebra of  $G$ ), then there is a unique solution  $u \in C([0, T], H^s(S^1)/\mathfrak{g})$  for some  $T > 0$  (which may be infinite). In our context this is a consequence of the fact that the geodesic equation is smooth, so that there is a unique solution  $\eta \in C^\infty([0, T], \operatorname{Diff}^s(S^1)/G)$  with  $\eta(0) = \operatorname{id}$  and  $\dot{\eta}(0) = u_0$ . Loss of smoothness of  $u$  in time occurs due to the fact that composition required to get  $u = \dot{\eta} \circ \eta^{-1}$  is not smooth. This

approach to the Euler equations was originally due to Ebin-Marsden [18]; for the Wunsch equation it was proved by Escher-Kolev-Wunsch [25], while for the Euler-Weil-Petersson equation it was proved by Escher-Kolev [26]. Castro-Córdoba [8] showed that if  $u_0$  is initially odd, then solutions to the Wunsch equation blow up in finite time; the authors of [4] extended this result to some data without odd symmetry. For the Euler-Weil-Petersson equation, it was not known whether initially smooth data would remain smooth for all time. However Gay-Balmaz and Ratiu [31] interpreted the equation in  $H^{3/2}$  as a **strong** Riemannian metric on a certain manifold and concluded that the velocity field  $u$  remains in  $H^{3/2}(S^1)$  for all time. We strengthen this to obtain a uniform  $C^1$  bound, which then by bootstrapping gives uniform bounds on all higher Sobolev norms and thus smoothness.

The main theorems of this paper settle the global existence question for the degenerate  $\dot{H}^r$  metrics corresponding to  $r = \frac{1}{2}$  (the Wunsch equation) and  $r = \frac{3}{2}$  (the Euler-Weil-Petersson equation).

**Theorem 10.** *Suppose  $s > \frac{3}{2}$  and  $u_0$  is an  $H^s$  velocity field on  $S^1$  with mean zero (i.e.,  $u_0 \in H^s(S^1)/\mathbb{R}$ ). Then the solution  $u(t)$  of the Wunsch equation with  $u(0) = u_0$  blows up in finite time.*

**Theorem 11.** *Suppose  $s > \frac{3}{2}$  and  $u_0$  is an  $H^s$  velocity field on  $S^1$ , and that the Fourier series of  $u_0$  has vanishing  $n = 0$ ,  $n = 1$ , and  $n = -1$  component; i.e.,  $u_0 \in H^s(S^1)/\mathfrak{sl}_2(\mathbb{R})$ . Then the solution  $u(t)$  of the Euler-Weil-Petersson equation with  $u(0) = u_0$  remains in  $H^s$  for all time. In particular if  $u_0$  is  $C^\infty$  then so is  $u(t)$  for all  $t > 0$ .*

Additionally, Theorem 12 almost immediately gives us that every mean zero solution of the Constantin-Lax-Majda equation [10] blows up in finite time. Overall, these two Theorems mean that the case  $r = \frac{3}{2}$  behaves the same as the cases for  $r > \frac{3}{2}$ , while the case  $r = \frac{1}{2}$  behaves the same as for  $r = 1$  (since all solutions of the Hunter-Saxton equation blow up in finite time [46]). We may conjecture that there is a critical value  $r_0$  such that for  $r > r_0$  all smooth mean-zero solutions remain smooth for all time, while for  $r < r_0$  all smooth mean-zero solutions blow up in finite time. Our guess is that  $r_0 = \frac{3}{2}$ , but the current method does not prove this; furthermore we do not know

what happens with geodesics for  $\frac{1}{2} < r < 1$  or  $1 < r < \frac{3}{2}$  even in the degenerate case.

Both equations arise naturally in the study of universal Teichmüller spaces. The Euler-Weil-Petersson equation was derived in [31] as the Euler-Arnold equation arising from the Weil-Petersson metric on the universal Teichmüller space. This geometry has been studied extensively by Takhtajan-Teo [80]; in particular they constructed the Hilbert manifold structure that makes Weil-Petersson a strong Hilbert metric (thus ensuring that geodesics exist globally). The Weil-Petersson geometry is well-known: the sectional curvature is strictly negative, and it is a Kähler manifold with almost complex structure given by the Hilbert transform. See Tromba [79] and Yamada [84] for further background on the Weil-Petersson metric on the universal Teichmüller space.

The Wunsch equation arises from the Riemannian metric  $\langle u, u \rangle = \int_{S^1} u H u_\theta dx$ , which is called the Velling-Kirillov metric and was proposed as a metric on the universal Teichmüller curve by Teo [77][78]. The Velling-Kirillov geometry was originally studied by Kirillov-Yur'ev [41]; although the sectional curvature is believed to be always positive, this is not yet proved. Furthermore the geometries are related in the sense that integrating the square of the symplectic form for the W-P geometry gives the symplectic form for the V-K geometry. Yet the properties of these geometries seem to be opposite in virtually every way: from Fredholmness of the exponential map [53][4] to the sectional curvature to the global properties of geodesics mentioned above.

**Theorem 12.** *Suppose  $s > \frac{3}{2}$  and  $u_0$  is an  $H^s$  velocity field on  $S^1$  with mean zero (i.e.,  $u_0 \in H^s(S^1)/\mathbb{R}$ ). Then the solution  $u(t)$  of the Wunsch equation with  $u(0) = u_0$  blows up in finite time.*

**Theorem 13.** *Suppose  $s > \frac{3}{2}$  and  $u_0$  is an  $H^s$  velocity field on  $S^1$ , and that the Fourier series of  $u_0$  has vanishing  $n = 0$ ,  $n = 1$ , and  $n = -1$  component; i.e.,  $u_0 \in H^s(S^1)/\mathfrak{sl}_2(\mathbb{R})$ . Then the solution  $u(t)$  of the Euler-Weil-Petersson equation with  $u(0) = u_0$  remains in  $H^s$  for all time. In particular if  $u_0$  is  $C^\infty$  then so is  $u(t)$  for all  $t > 0$ .*

### 2.1.1 Proof of the Main Theorems

#### 2.1.1.1 Rewriting the Equations and Proof of Theorem 12

Let us first sketch the blowup argument for the Wunsch equation from [4], which extended the argument of Castro-Córdoba [8]. The Wunsch equation is given for mean-zero vector fields  $u$  on  $S^1$  (identified with functions) by the formula

$$\omega_t + u\omega_\theta + 2u_\theta\omega = 0, \quad \omega = Hu_\theta. \quad (2.4)$$

In terms of the Lagrangian flow  $\eta$  given by (2.3), we may rewrite this as

$$\partial_t \omega(t, \eta(t, \theta)) + 2\eta_{t\theta}(t, \theta)\omega(t, \eta(t, \theta)) / \eta_\theta(t, \theta) = 0$$

which leads to the conservation law

$$\eta_\theta(t, \theta)^2 \omega(t, \eta(t, \theta)) = \omega_0(\theta). \quad (2.5)$$

Applying the Hilbert transform to both sides of (2.4) and using the following Hilbert transform identities (valid for mean-zero functions  $f$ ):

$$H(Hf) = -f \quad \text{and} \quad 2H(fHf) = (Hf)^2 - f^2, \quad (2.6)$$

one obtains [4] an equation for  $u_\theta = -H\omega$ :

$$u_{t\theta} + uu_{\theta\theta} + u_\theta^2 = -F + \omega^2 \quad (2.7)$$

where the function  $F$  is a spatially nonlocal force given for each fixed time  $t$  by

$$F = -uu_{\theta\theta} - H(uHu_{\theta\theta}). \quad (2.8)$$

In Lagrangian form, using the conservation law equation (2.7) becomes

$$\eta_{tt\theta}(t, \theta) = \frac{\omega_0(\theta)^2}{\eta_\theta(t, \theta)^3} - F(t, \eta(t, \theta))\eta_\theta(t, \theta). \quad (2.9)$$

It follows that if there is a point  $\theta_0$  such that  $u'(\theta_0) < 0$  and  $\omega_0(\theta_0) = 0$ , then we will have  $\eta_\theta(0, \theta_0) = 1$ ,  $\eta_{t\theta}(0, \theta_0) < 0$ , and  $\eta_{tt\theta}(t, \theta_0) < 0$  for all  $t$ , so that  $\eta_\theta(t, \theta_0)$  must reach zero in finite

time (which leads to  $u_\theta \rightarrow -\infty$ ). Our proof that **all** solutions blow up consists of showing that this condition happens for **every** initial condition  $u_0$  with  $\omega_0 = Hu_0$ .

*Proof of Theorem 12.* From the discussion above, the proof reduces to proving the following statement. Suppose  $f: S^1 \rightarrow \mathbb{R}$  is a smooth function with mean zero, and let  $g = Hf$ . Then there is a point  $\theta_0 \in S^1$  with  $f'(\theta_0) < 0$  and  $g'(\theta_0) = 0$ .

Let  $p$  be the unique harmonic function in the unit disc  $\mathbb{D}$  such that  $p|_{S^1} = f$ , and let  $q$  be its harmonic conjugate normalized so that  $q|_{S^1} = g$ . Then in polar coordinates we have the Cauchy-Riemann equations

$$rp_r(r, \theta) = q_\theta(r, \theta) \quad \text{and} \quad rq_r(r, \theta) = -p_\theta(r, \theta), \quad (2.10)$$

and we have  $p(1, \theta) = f(\theta)$  and  $q(1, \theta) = g(\theta)$ .

Since  $q$  is harmonic, its maximum value within  $\mathbb{D}$  occurs on the boundary  $S^1$  at some point  $\theta_0$ . The maximum of  $g$  occurs at the same point, so that  $g'(\theta_0) = 0$ . By the Hopf lemma, we have  $q_r(1, \theta_0) > 0$ , so equations (2.10) imply that  $f'(\theta_0) = p_\theta(1, \theta_0) < 0$ .  $\square$

**Remark.** *This argument also works when the domain is  $\mathbb{R}$  and the functions have suitable decay conditions imposed. It can thus be applied to demonstrate that **every** mean zero solution of the Constantin-Lax-Majda equation [10]*

$$\omega_t - v_x \omega = 0, \quad v_x = H\omega$$

*blows up in finite time, using the same argument as in that paper via the explicit solution formula.*

Now let us rewrite the Euler-Weil-Petersson equation to obtain the analogue of formula (2.7). Recall from the introduction that it is given explicitly by

$$\omega_t + u\omega_\theta + 2u_\theta\omega = 0, \quad \omega = -Hu_{\theta\theta\theta} - Hu_\theta. \quad (2.11)$$

**Proposition 2.1.1.** *The Euler-Weil-Petersson equation (2.11) is equivalent to the equation*

$$u_{t\theta} = H(uHu_{\theta\theta}) + H(1 + \partial_\theta^2)^{-1} [2u_\theta Hu_\theta - u_{\theta\theta} Hu_{\theta\theta}], \quad (2.12)$$

In terms of the Lagrangian flow (2.3), equation (2.12) takes the form

$$\frac{\partial}{\partial t} u_\theta(t, \eta(t, \theta)) = -F(t, \eta(t, \theta)) + G(t, \eta(t, \theta)) \quad (2.13)$$

where  $F$  is defined by formula (2.8) and  $G$  is given by

$$G = H(1 + \partial_\theta^2)^{-1} [2u_\theta H u_\theta - u_{\theta\theta} H u_{\theta\theta}]. \quad (2.14)$$

Here the operator  $(1 + \partial_\theta^2)$  is restricted to the orthogonal complement of the span of  $\{1, \sin \theta, \cos \theta\}$  so as to be invertible.

*Proof.* Equation (2.11) may be written

$$-H(1 + \partial_\theta^2)u_{t\theta} = (1 + \partial_\theta^2)(uH u_{\theta\theta}) - u_{\theta\theta} H u_{\theta\theta} + 2u_\theta H u_\theta,$$

using the product rule. We now solve for  $u_{t\theta}$  by applying  $H$  to both sides and inverting  $(1 + \partial_\theta^2)$ . To do this, we just need to check that the term  $(2u_\theta H u_\theta - u_{\theta\theta} H u_{\theta\theta})$  is orthogonal to the subspace spanned by  $\{1, \sin \theta, \cos \theta\}$ . In fact this is true for every function  $fHf$  when  $f$  is  $2\pi$ -periodic with mean zero, since the formulas (2.6) imply both that  $fHf$  has mean zero and that it has period  $\pi$ .

The only additional thing happening in equation (2.13) is the chain rule formula

$$\partial_t u_\theta(t, \eta(t, \theta)) = u_{t\theta}(t, \eta(t, \theta)) + u_{\theta\theta}(t, \eta(t, \theta))\eta_t(t, \theta) = (u_{t\theta} + uu_{\theta\theta})(t, \eta(t, \theta)).$$

□

To prove Theorem 13, we want to show that  $\|u_\theta\|_{L^\infty}$  remains bounded for all time, and by formula (2.13) it is sufficient to bound both  $\|F\|_{L^\infty}$  and  $\|G\|_{L^\infty}$ . We will do this in the next Section.

### 2.1.2 The Bounds on $F$ and $G$

In [4], it was shown that the function  $F$  given by (2.8) is positive for any mean-zero function  $u: S^1 \rightarrow \mathbb{R}$ . This is essential for proving blowup for the Wunsch equation.

**Theorem 14** (Bauer-Kolev-Preston). *Let  $u: S^1 \rightarrow \mathbb{R}$  be a function with Fourier series  $u(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$  with  $c_0 = 0$ . If  $\Lambda = H\partial_\theta$  so that  $\Lambda(e^{in\theta}) = |n|e^{in\theta}$ , and if  $g_p = H(uH\Lambda^p u) + u\Lambda^p u$  for a positive number  $p$ , then for every  $\theta \in S^1$  we have*

$$g_p(\theta) = 2 \sum_{k=1}^{\infty} [k^p - (k-1)^p] |\phi_k(x)|^2, \quad \text{where } \phi_k(\theta) = \sum_{m=k}^{\infty} c_m e^{im\theta} \quad (2.15)$$

*In particular  $F = -uu'' - H(uHu'')$  is positive at every point if  $u$  is not constant.*

Another perspective on the positivity of  $F$  is discussed in Silvestre-Vicol [72]. There, while studying a slightly different version of the generalized Constantin-Lax-Majda equation over  $\mathbb{R}$ , they demonstrated that  $F$  can also be represented as

$$F(t, 0) = \left\| \frac{u(x) - u(0)}{x} \right\|_{\dot{H}^{1/2}(\mathbb{R})}^2, \quad (2.16)$$

This insight into the structure of  $F$  helps explain the positivity result of the previous Theorem. We would now like to bound  $F$  in terms of  $\|u\|_{\dot{H}^{3/2}}^2 := \int_{S^1} (Hu)(u''' + u') d\theta$ . It is possible to do so using results similar to formula (2.16) on the circle, however the simplest proof uses the Fourier series formula (2.15) directly.

**Theorem 15.** *Let  $u: S^1 \rightarrow \mathbb{R}$  be a smooth function with Fourier coefficients  $c_n$  such that  $c_0 = c_1 = c_{-1} = 0$ , and let  $F = -uu'' - H(uHu'')$ . Then for every  $\theta \in S^1$ , we have*

$$F(\theta) \leq C \|u\|_{\dot{H}^{3/2}}^2, \quad (2.17)$$

where  $\|u\|_{\dot{H}^{3/2}(S^1)}^2 = \int_{S^1} (Hu)(u''' + u') d\theta$  and  $C$  is a constant independent of  $u$ .

*Proof.* By equation (2.15) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} F(\theta) &= \sum_{n=1}^{\infty} (2n-1) \left| \sum_{m=n}^{\infty} c_m \right|^2 \leq \sum_{n=1}^{\infty} (2n-1) \sum_{m=n}^{\infty} m(m-1) |c_m|^2 \sum_{m=n}^{\infty} \frac{1}{m(m-1)} \\ &\leq \sum_{n=1}^{\infty} \frac{2n-1}{n} \sum_{m=n}^{\infty} m^2 |c_m|^2 \leq 2 \sum_{m=1}^{\infty} \sum_{n=1}^m m^2 |c_m|^2 \leq 2 \sum_{m=1}^{\infty} m^3 |c_m|^2. \end{aligned}$$

□

Note that  $G$  given by (2.14) consists of two similar terms, and the following Theorem takes care of both at the same time as a consequence of Hilbert's double series inequality.

**Theorem 16.** *Suppose  $f: S^1 \rightarrow \mathbb{R}$  is a smooth function and that  $g = H(1 + \partial_\theta^2)^{-1}(f'Hf')$ . Then  $\|g\|_{L^\infty} \leq 4\pi\|f\|_{\dot{H}^{1/2}}^2$ .*

*Proof.* Expand  $f$  in a Fourier series as  $f(\theta) = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}$ , and let  $h = f'Hf'$ . Then we have

$$f'Hf'(\theta) = i \sum_{m, n \in \mathbb{Z}} mn f_m f_n (\text{sign } n) e^{i(m+n)\theta} = i \sum_{k \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |n| (k-n) f_{k-n} f_n \right) e^{ik\theta} = i \sum_{k \in \mathbb{Z}} h_k e^{ik\theta},$$

where

$$h_k = \sum_{n \in \mathbb{Z}} |n| (k-n) f_{k-n} f_n.$$

Now let us simplify  $h_k$ : we have for  $k > 0$  that

$$\begin{aligned} h_k &= \sum_{n=1}^{\infty} n(k-n) f_n f_{k-n} + \sum_{n=1}^{\infty} n(k+n) \overline{f_n} f_{k+n} \\ &= \sum_{n=1}^{k-1} n(k-n) f_n f_{k-n} + \sum_{m=1}^{\infty} (k+m)(-m) f_{k+m} \overline{f_m} + \sum_{n=1}^{\infty} n(k+n) \overline{f_n} f_{k+n}, \end{aligned}$$

where we used the substitution  $m = n - k$ . Clearly the middle term cancels the last term, so

$$h_k = \sum_{n=1}^{k-1} n(k-n) f_n f_{k-n}. \quad (2.18)$$

It is easy to see that  $h_0 = 0$  due to cancellations, while if  $k < 0$ , we get

$$h_k = - \sum_{n=1}^{|k|-1} n(|k|-n) \overline{f_n f_{|k|-n}} = -\overline{h_{|k|}}.$$

Note in particular that  $h_1 = h_{-1} = 0$ . We thus obtain

$$f'Hf'(\theta) = \sum_{k=2}^{\infty} (ih_k e^{ik\theta} - i\overline{h_k} e^{-ik\theta}),$$

so that

$$H(f'Hf')(\theta) = \sum_{k=2}^{\infty} h_k e^{ik\theta} + \overline{h_k} e^{-ik\theta} = 2\text{Re} \left( \sum_{k=2}^{\infty} h_k e^{ik\theta} \right).$$

It now makes sense to apply  $(1 + \partial_\theta^2)^{-1}$  to this function, and we obtain

$$g(\theta) = 2\text{Re} \left( \sum_{k=2}^{\infty} \frac{h_k}{1-k^2} e^{ik\theta} \right),$$

so that

$$\begin{aligned}
\|g\|_{L^\infty} &\leq 2 \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} \frac{n(k-n)|f_n||f_{k-n}|}{k^2-1} = 2 \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \frac{n(k-n)|f_n||f_{k-n}|}{k^2-1} \\
&= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{nm|f_n||f_m|}{(n+m)^2-1} \leq 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\sqrt{nm}|f_n||f_m|}{n+m} \\
&\leq 4\pi \left( \sum_{n=1}^{\infty} n|f_n|^2 \right) = 4\|f\|_{\dot{H}^{1/2}(S^1)}^2,
\end{aligned}$$

where the inequality in the last line is precisely the well-known Hilbert double series theorem ([34], Section 9.1).  $\square$

Applying this Theorem to the terms in (2.14), we obtain the following straightforward Corollary which takes care of the second term in the equation (2.13) for  $u_\theta$  in the Euler-Weil-Petersson equation.

**Corollary 2.1.1.** *Suppose  $u$  is vector field on  $S^1$ , and let  $G = H(1 + \partial_\theta^2)^{-1}[2u_\theta H u_\theta - u_{\theta\theta} H u_{\theta\theta}]$  as in (2.14). Then we have*

$$\|G\|_{L^\infty} \leq 8\pi \|u\|_{\dot{H}^{1/2}(S^1)}^2 + 4\pi \|u\|_{\dot{H}^{3/2}(S^1)}^2, \quad (2.19)$$

in terms of the degenerate seminorm  $\|u\|_{\dot{H}^{3/2}(S^1)}^2 = \int_{S^1} (Hu)(u''' + u') d\theta$ .

### 2.1.2.1 Proof of Theorem 13

The work of Escher and Kolev shows that solutions of (2.11) are global as long as we can control the  $C^1$  norm  $\|u\|_{C^1(S^1)}$ . This follows from the no-loss/no-gain Lemma 4.1 of [26] and the general estimate for Sobolev  $H^q$  norms in terms of  $C^1$  norms from Theorem 5.1 of [27].

**Theorem 17** (Escher-Kolev). *Let  $u$  be a smooth solution of (2.11) on a maximal time interval  $[0, T)$ . If there is a constant  $C$  such that  $\|\partial_\theta u(t, \theta)\|_{L^\infty(S^1)} \leq C(1+t)$  for all  $t \in [0, T)$ , then in fact  $T = \infty$ .*

Hence all we need to do is obtain a uniform bound for the  $C^1$  norm of  $u$ . Since the  $\dot{H}^{3/2}$  seminorm of a solution of (2.11) is constant by energy conservation, it is sufficient to bound the  $C^1$

norm in terms of the  $\dot{H}^{3/2}$  seminorm. Note that the  $H^{3/2}(S^1)$  norm does not in general control the  $C^1(S^1)$  norm of an arbitrary function  $f$  on  $S^1$ ; we need to use the special structure of the equation (2.11) to get this.

*Proof of Theorem 13.* Proposition 2.1.1 shows that

$$\frac{d}{dt}\|u_\theta\|_{L^\infty} \leq \|F\|_{L^\infty} + \|G\|_{L^\infty}.$$

Using Theorem 15, we obtain  $\|F\|_{L^\infty} \leq CE_0$  where  $E_0 = \|u\|_{\dot{H}^{3/2}}^2$ , which is constant in time since  $u$  is an Euler-Arnold equation. Similarly Corollary 2.1.1 yields  $\|G\|_{L^\infty} \leq C\|u_\theta\|_{\dot{H}^{1/2}}^2 + CE_0$ . Since  $u$  is always chosen as the representative of the equivalence class that has  $c_1 = c_{-1} = 0$  (i.e., its Fourier coefficients are only nonzero for  $|n| \geq 2$ ), we can easily bound both of these lower-order terms above by some constant multiple of  $E_0$ .

It follows that  $\frac{d}{dt}\|u_\theta\|_{L^\infty} \leq CE_0$ , so that  $\|\partial_\theta u(t)\|_{L^\infty} \leq \|\partial_\theta u_0\|_{L^\infty} + CE_0 t$ , which shows that  $u_\theta$  cannot approach infinity in finite time. This proves that the solution of the EWP equation (2.11) remains in any  $H^s$  space that  $u_0$  begins in for any  $s > \frac{3}{2}$ , using Theorem 17 and the no-loss/no-gain Lemma 4.1 of [26].  $\square$

## 2.2 The Conformal Welding Picture of Geodesic Blowup

In Teichmüller theory there is a natural identification between diffeomorphisms of the unit circle and curves in the plane, up to certain normalizations, given by conformal welding. Here we use numerical simulations to map the geodesics corresponding to the Wunsch and EWP equations to their respective spaces of curves in  $\mathbb{C}$ , which then correspond to spaces of appropriately normalized conformal maps of the unit disk. It would be interesting to use this alternative representation of geodesics to prove the results of Theorem 12 directly, since it could generate a new approach to other Euler-Arnold equations with similar geometric properties for which these results are not known, such as the surface quasigeostrophic equation [81] and the 3D Euler equation [66].

### 2.2.1 Conformal Welding on $T(1)$ and $\mathcal{T}(1)$

We really have two different situations where we may apply conformal welding: for the Universal Teichmüller Space,  $T(1)$ , and for the Universal Teichmüller Curve,  $\mathcal{T}(1)$ .

#### 2.2.1.1 $T(1)$

As is discussed in Sharon and Mumford [70], for the Universal Teichmüller Space  $T(1)$ , conformal welding may be thought of as a map:

$$\Gamma : \mathcal{C} \rightarrow \text{Diff}(S^1)/\text{PSL}_2(\mathbb{R}),$$

where  $\mathcal{C}$  consists of equivalence classes of smooth, closed, simple curves in  $\mathbb{C}$  modulo scaling and translation operations. Given a representative  $C$  of an equivalence class  $[C] \in \mathcal{C}$ , by the Riemann Mapping Theorem, there exists a univalent, holomorphic function  $\Phi_- : \mathbb{D} \rightarrow \mathbb{C}$  such that  $\partial\Phi_-(\mathbb{D}) = C$ . This function is unique up to precomposition by an element of  $\text{PSL}_2(\mathbb{R})$ . Then, we also find a (this time unique) holomorphic map  $\Phi_+ : \mathbb{D}^* \rightarrow \mathbb{C}$ , where  $\mathbb{D}^*$  is the exterior of the unit disk, such that  $\partial\Phi_+(\mathbb{D}^*) = C$ ,  $\Phi_+(\infty) = \infty$  and  $\Phi'_+(\infty) > 0$ .

We then define

$$\Gamma : [C] \mapsto [\eta]_{\text{PSL}_2(\mathbb{R})} \ni \eta = \Phi_+^{-1} \circ \Phi_-|_{S^1}$$

The fact that this map is bijective follows from the construction of its inverse, see Sharon and Mumford [70], or Lehto [45]. One may construct the inverse by solving a Fredholm integral equation of the second kind as in Feiszli and Mumford [29]. One must solve:

$$K(F) + F = e^{i\theta} \tag{2.20}$$

where  $F(\theta) = \Phi_+(e^{i\theta})$  and

$$K(F)(\theta) = \frac{i}{2} \int_{S^1} \left( \cot \left( \frac{\theta - \phi_1}{2} \right) - (\eta^{-1})_x(\phi_1) \cot \left( \frac{\eta^{-1}(\theta) - \eta^{-1}(\phi_1)}{2} \right) \right) F(\phi_1) d\phi_1.$$

The results of Feiszli and Mumford [29] ensure that this has a unique solution. Then we have a practical way of computing

$$\Gamma^{-1} : \eta \mapsto F([0, 2\pi)).$$

### 2.2.1.2 $\mathcal{T}(1)$

Alternatively, as is discussed in Kirillov [40] and Teo [77], one may consider conformal welding for the Universal Teichmüller curve,  $\mathcal{T}(1)$ , which may be thought of as a map:

$$\tilde{\Gamma} : \tilde{\mathcal{C}} \rightarrow \text{Diff}(S^1)/\text{Rot}(S^1),$$

where  $\tilde{\mathcal{C}}$  consists of smooth, closed, simple curves in  $\mathbb{C}$  with conformal radius one at the origin. We then proceed in a similar fashion as before. This time, when we find a univalent, holomorphic  $\Phi_- : \mathbb{D} \rightarrow \mathbb{C}$  such that  $\partial\Phi_-(\mathbb{D}) = C$ , we also demand that  $\Phi_-$  be the unique mapping such that  $\Phi_-(0) = 0$  and  $\Phi'_-(0) > 0$  (which is the normalization demanded by the Riemann Mapping Theorem). Since  $C$  has conformal radius one,  $\Phi'_-(0) = 1$ . We find  $\Phi_+$  in the same way as before, and then have the same definition for  $\tilde{\Gamma}$  as for  $\Gamma$ :

$$\tilde{\Gamma} : C \mapsto [\eta]_{\text{Rot}(S^1)} \ni \eta = \Phi_+^{-1} \circ \Phi_-|_{S^1}$$

where the equivalence class is now taken to be in  $\text{Diff}(S^1)/\text{Rot}(S^1)$  as opposed to  $\text{Diff}(S^1)/\text{PSL}_2(\mathbb{R})$  in the case of  $\Gamma$ . That this map is bijective is proved in Kirillov [40] for smooth curves and in Teo [77] for the more general situation of quasicircles. As mentioned above, our goal was to use numerical simulations to take the Lagrangian trajectories corresponding to the Wunsch equation, and map them (at each time  $t$ ) via  $\tilde{\Gamma}^{-1}$ , to  $\tilde{\mathcal{C}}$ . We employed the integral kernel method from Feiszli and Mumford [29]. At the moment, our  $[\eta]_{\text{Rot}(S^1)}$  will be mapped via this method to the correct curve in  $\mathbb{C}$  up to scaling and translation. The curve can be readily normalized so that it encloses 0 and has conformal radius 1 (i.e. so that it lies within  $\tilde{\mathcal{C}}$ ). Of course, the issue is that there is a whole family of possible such normalized curves within an equivalence class in  $\mathcal{C}$ .

This issue can be readily solved by observing the following. Suppose that  $C_1, C_2 \in \tilde{\mathcal{C}}$  such that  $a(C_1 + v) = C_2$  for some  $v \in \mathbb{C}$ , and  $a \in \mathbb{R}^+$ . Let  $f_1 : \mathbb{D} \rightarrow \mathbb{C}$  and  $f_2 : \mathbb{D} \rightarrow \mathbb{C}$  be the corresponding

Riemann mappings with boundary values  $C_1$  and  $C_2$  respectively, with  $f_i(0) = 0$  and  $f_i'(0) = 1$ . Then there is a unique element  $\phi \in \mathrm{PSL}_2(\mathbb{R})$  such that  $a(f_1 \circ \phi + v) = f_2$ . This explains the differences in normalizations between the  $T(1)$  and  $\mathcal{T}(1)$  situations and now motivates the technique for our numerical simulation computing  $\tilde{\Gamma}^{-1}$ . First, we take our geodesic  $\eta(t)$  corresponding to the Wunsch equation in Lagrangian coordinates. We employ the algorithm in Feiszli and Mumford [29] and solve (2.20) to obtain the curve we want up to scaling and translation. We take any normalization (up to translation and scaling) of this curve so that it lies in  $\tilde{C}$ . We then find  $\Phi_+$  and  $\Phi_-$  and compute  $\zeta = (\Phi_+)^{-1} \circ \Phi_-|_{S^1}$ . By previous arguments we have  $\zeta \in [\eta]_{\mathrm{PSL}_2(\mathbb{R})}$ . Hence there exists a  $\phi \in \mathrm{PSL}_2(\mathbb{R})/\mathrm{Rot}(S^1)$  such that  $\zeta \circ \phi|_{S^1} = \eta$ . Hence we compute  $\phi|_{S^1} = \zeta^{-1} \circ \eta$ . Then we extend  $\phi|_{S^1}$  to  $\mathbb{D}$  (simply by observing that we only need to know 3 values of  $\phi$  to know  $\phi$  itself), and then we compute  $\phi(0) = (f_1)^{-1}(-v)$ . This yields the correct curve in  $\tilde{C}$ .

### 2.3 Numerical Simulations

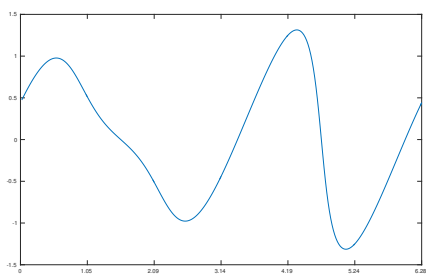
In this section we show the results of numerical simulations solving the Wunsch and Euler-Weil-Petersson equations, as well as implement the conformal welding process above. Here we will:

- Solve the Wunsch and Euler-Weil-Petersson equations in Eulerian coordinates.
- Solve the flow equation to switch to Lagrangian coordinates
- Plot the Ermakov Pinney trajectories of each equation
- Map the geodesics to the space of univalent holomorphic functions with the appropriate normalizations ( $\mathcal{D}$  or  $\tilde{\mathcal{D}}$ ) via conformal welding.
- Undo the conformal welding map via the Schwarz-Christoffel formula (as is done in Sharon and Mumford [70])

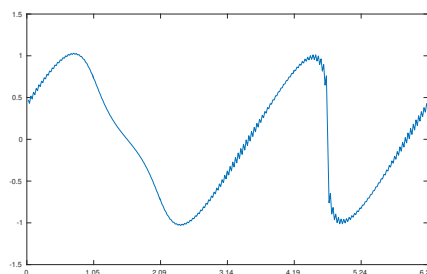
### 2.3.1 Solutions to EWP and Wunsch

Here we implemented a Fourier-Galerkin method to get a system of ODEs, coupled with a 4th order Runge-Kutta method to solve each ODE that arises. The following is a collection of solutions for the EWP and Wunsch equations with initial condition  $u_0(x) = \sin(2x)$ . For each equation we have  $t_0 = 0$  and  $t_{fin} = .5$ .

Figure 2.1: Eulerian Solutions to Wunsch with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ . Note that the slopes approach  $-\infty$ ; after this the numerical solution appears to become singular everywhere simultaneously. It is not clear if this is what actually happens.

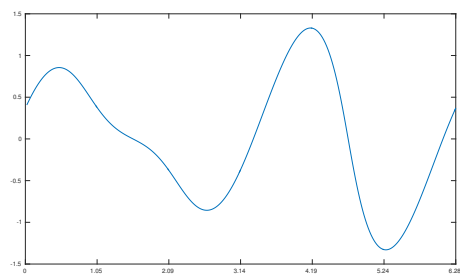


t=.125 (before blowup)

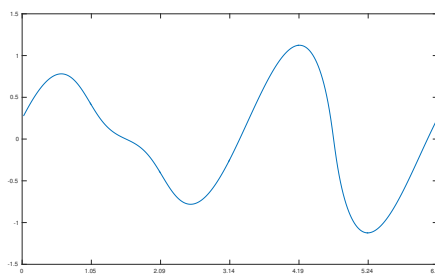


t=.25 (after blowup)

Figure 2.2: Eulerian Solutions to EWP with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ . The profile steepens but does not become singular.

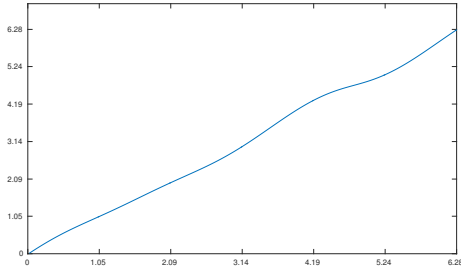


t=.25

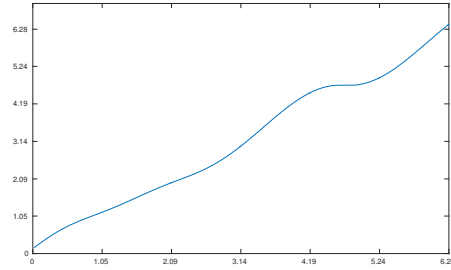


t=.5

Figure 2.3: Lagrangian Solutions to Wunsch with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ . As  $u_\theta$  approaches  $-\infty$ , the slope of  $\eta$  approaches zero, and  $\eta$  leaves the diffeomorphism group.

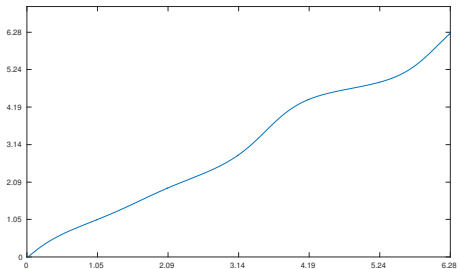


$t=.125$  (before blowup)

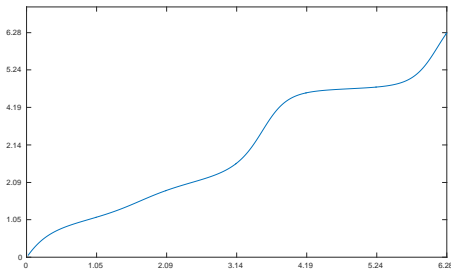


$t=.25$  (after blowup)

Figure 2.4: Lagrangian Solutions to EWP with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ . It appears that  $\eta$  is flattening substantially, but the slope still remains positive.



$t=.25$



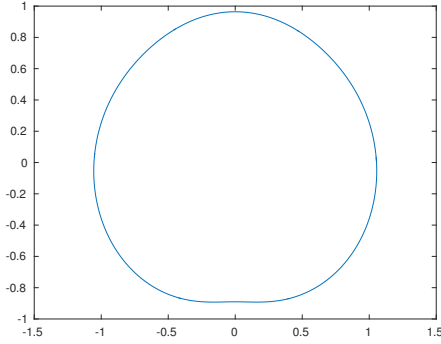
$t=.5$

### 2.3.2 Conformal Welding

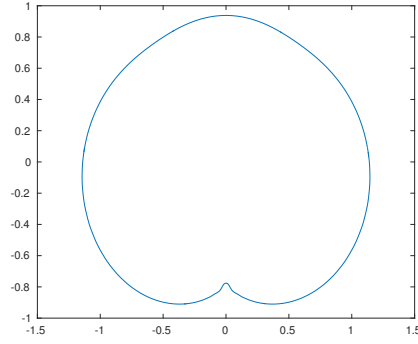
Here we used Mumford's MATLAB code [56] to solve (2.20). For EWP solutions, this yields a representative of the corresponding equivalence class in  $\mathcal{C}$ . For solutions to the Wunsch equation, we proceed by normalizing using the technique discussed above. As in Sharon and Mumford [70], we employed the Schwarz-Christoffel map, as implemented in Tobin Driscoll's code available at the website <http://www.math.udel.edu/~driscoll/SC/index.html> to construct the Riemann mappings and hence diffeomorphisms associated to each curve. The first thing we note about the Welding curves is that for the above Wunsch solutions, the translation and scaling operations as

discussed above are essentially trivial. We believe that this has to do with the fact that the  $\sin(x)$  Fourier mode in the above solution to the Wunsch equation is zero. This at least makes heuristic sense as the bulk of the translating and dilating should happen in the  $\mathrm{PSL}_2(\mathbb{R})$  fibre, which in this case is zero. One can obtain small amounts of shifting and translating by making sure that the initial data has either  $\sin(x)$  or  $\cos(x)$  terms. Second, we note the similarity in the shapes of the solutions. Again, this is perhaps not surprising as the  $H^{3/2}$  metric is obtained by averaging over the fibres of the  $H^{1/2}$  metric as in Teo [77].

Figure 2.5: Trajectories in  $\tilde{\mathcal{C}}$  after welding solutions to the Wunsch equation with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ .

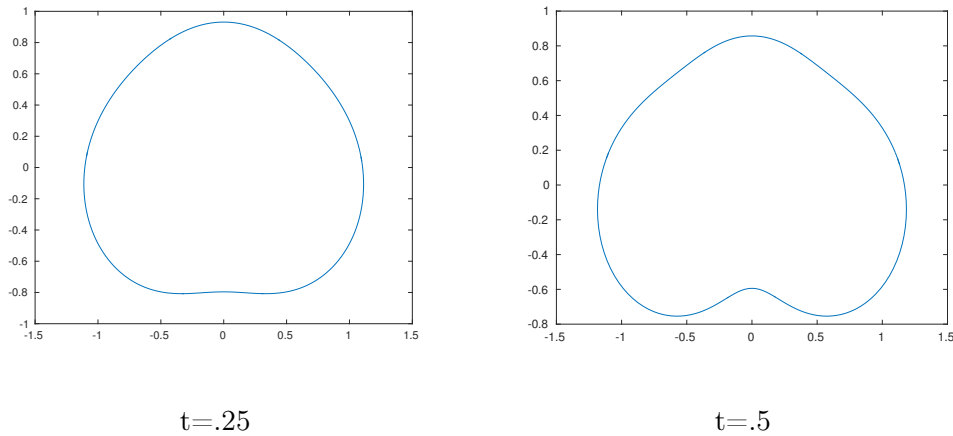


t=.125



t=.25 (after blowup)

Figure 2.6: Trajectories in  $\mathcal{C}$  after welding solutions to EWP with  $u_0 = \sin(2x) + \frac{1}{2} \cos(3x)$ . The change in concavity seems to remain smooth, with continuous tangents.



### 2.3.3 Inverting the Welding Map

As in Sharon and Mumford [70], we employ the Schwarz-Christoffel map, as implemented in Tobin Driscoll's code available at the website <http://www.math.udel.edu/~driscoll/SC/index.html> to construct the Riemann mappings  $\Phi_+$  and  $\Phi_-$  associated to each curve. We then compute  $\Phi_+^{-1} \circ \Phi_-|_{S^1}$  explicitly. There is good agreement with the original diffeomorphisms.

Figure 2.7: Recovered diffeomorphisms after welding for the Wunsch equation

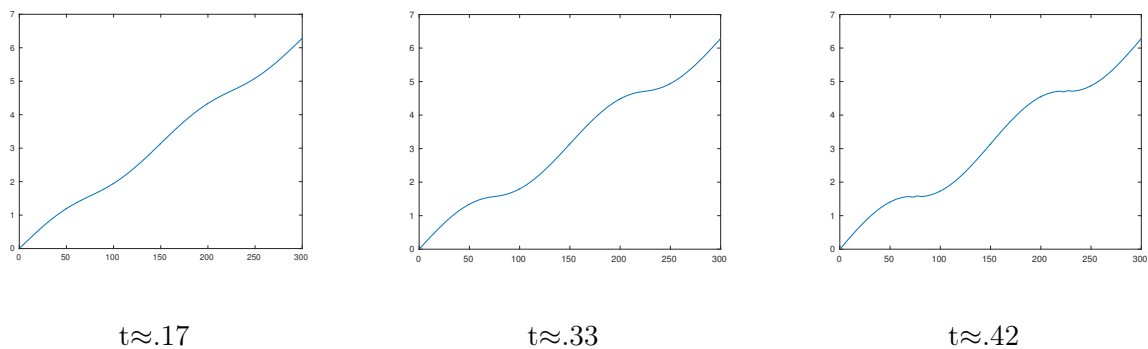
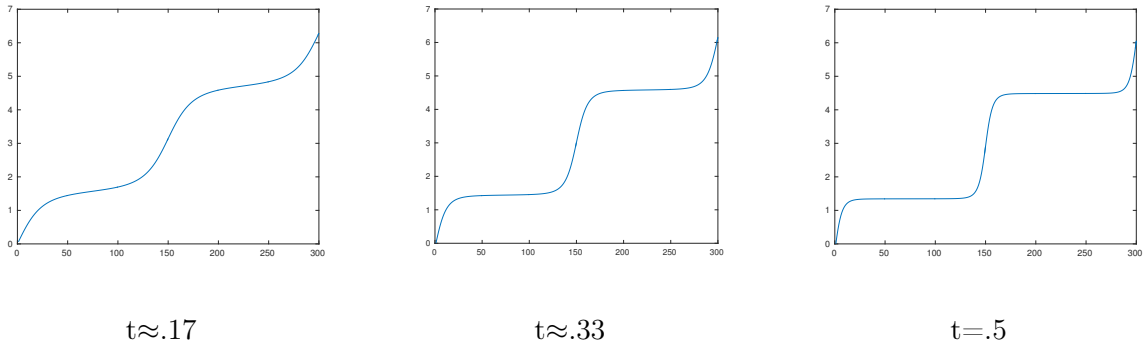


Figure 2.8: Recovered diffeomorphisms after welding for the EWP equation



## 2.4 Interpolating $L^2$ and $\dot{H}^1$

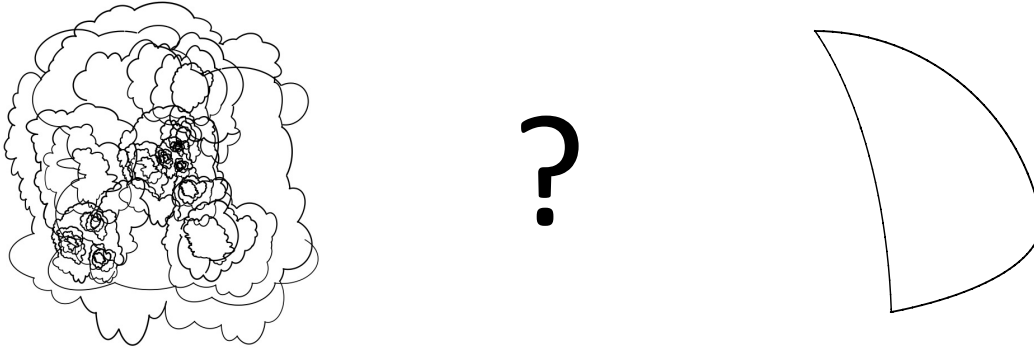
Both the right-invariant  $L^2$  and  $\dot{H}^1$  metrics on  $\mathcal{D}(S^1)$  have been studied in Constantin-Kolev [13], Lenells [46], and Khesin-Lenells-Misiołek-Preston [38]. So it makes sense to try and interpret the  $\dot{H}^{1/2}$  metric as lying between them and interpolating between their properties. See Figure 2.9. For example, it's known that both  $L^2$  and  $\dot{H}^1$  exhibit nonnegative sectional curvature. In all known cases computed the  $\dot{H}^{1/2}$  also has positive curvature, so it's currently a conjecture that it always does. See Chhay [7] for more details. In this section we begin an investigation into the global structure of conjugate points on  $\mathcal{D}(S^1)$  in the  $\dot{H}^{1/2}$  metric. Currently, all the work into this so far has been done in Bauer-Kolev-Preston [4].

It was shown in Preston [65] that on the 3D volumorphism group in the right-invariant  $L^2$  metric, if a geodesic (i.e., solution to the incompressible Euler equations) blows up, then there is a condition on the stretching matrix that is satisfied or the geodesic passes through what's called a **conjugate cascade** as it approaches the blow-up time. This is a sequence of points along the geodesic where each one is conjugate to the next, note that this is much stronger than a sequence of points conjugate to a single point. Intuitively, conjugate points are associated to positive curvature so this could happen if the geodesic passes through regions of rapidly increasing curvature. The diffeomorphism group may in some sense bunch up arbitrarily tightly as the geodesic is about to leave. Recall that from the introduction to this thesis that from an analytic perspective we're thinking of the Wunsch equation as mathematically similar to the incompressible Euler equations. So, it makes

sense that it should be similar geometrically. In Bauer-Kolev-Preston [4] the authors showed that in the  $\dot{H}^{1/2}$  metric,  $\mathcal{D}(S^1)$  has a non-Fredholm exponential map and unbounded curvature, both of which are properties of the 3D volumorphism group in the  $L^2$  metric.

The goal in this section was to show that as a geodesic in the  $\dot{H}^{1/2}$  metric approaches blowup, it must pass through a conjugate cascade as in the  $L^2$  metric in 3D. Unfortunately this has proven to be a difficult problem and I haven't been able to solve it. What follows has been my approach which will highlight what have been the key difficulties and possibilities towards a solution.

Figure 2.9: Interpolating  $L^2$  and  $\dot{H}^1$



$L^2$ : wildly pathological       $\dot{H}^{1/2}$ : somewhere in-between       $\dot{H}^1$ : a piece of a sphere!  
(artistic interpretation)

#### 2.4.1 The Structure of Conjugate Points in $L^2$ and $\dot{H}^1$

Again, since we can think of  $\mathcal{D}(S^1)$  in the  $\dot{H}^{1/2}$  metric as lying between  $L^2$  and  $\dot{H}^1$ , it makes sense to consider the structure of conjugate points on these spaces first. One can immediately see from Figure 2.9 that we shouldn't expect  $\dot{H}^1$  to have any conjugate points. The reason is simple: as an infinite-dimensional octant of a sphere, a geodesic will run off the space before it can come around to intersect any other geodesic. This computation was done explicitly in Khesin-Lenells-Misiołek-Preston [39].

The situation for the  $L^2$  metric is more complex. In general, its geometry has much more wild

behavior. For example, in Michor-Mumford [50] it was shown that its geodesic distance vanishes. That is, given any two points in  $\mathcal{D}(S^1)$  there exist geodesics of arbitrarily short length between them. One possible interpretation from Michor-Mumford [50] is that its curvature is heavily positive and results in the space curling in on itself arbitrarily tightly. In fact, the behavior of its conjugate points is equally if not even more troubling. In Disconzi-Ebin-Misiołek-Preston [16], it is shown that given a  $C^1$  velocity field  $u_0$  with a point  $x_0$  such that  $u_0(x_0) = 0$ , there exist corresponding solutions to the Jacobi equation that aren't  $C^1$ . This points to a potentially major difficulty in the geometric theory of the right-invariant  $L^2$  metric on  $\mathcal{D}(S^1)$ . In fact, even along positive  $u_0$ , geodesics can have a wildly pathological structure of its conjugate points.

**Theorem 18.** *On  $\mathcal{D}(S^1)$  in the right-invariant  $L^2$  metric, and for  $J = dL_\eta v$  for  $v \in T_{id}\mathcal{D}(S^1)$ , the left translated Morse index form (1.52) is given by*

$$I(J, J) = \int_a^b \int_{S^1} \eta_x^3 v_t^2 - u_0(v_t v_x - v_{xt} v) dx dt. \quad (2.21)$$

Moreover, for any  $u_0 \in T_{id}\mathcal{D}(S^1)$  such that  $u_0 > 0$ , there exists a  $v$  such that  $v(a) = v(b) = 0$  and  $I(J, J) < 0$ .

By lemma 1.2.1, this implies that conjugate points are in fact **dense** along a geodesic. As alluded to before this is highly pathological and troubling behavior for a space to have. With this in mind, we can see that the  $\dot{H}^{1/2}$  metric lies on a spectrum where at one end, with  $\dot{H}^1$  there are no conjugate points, where on the other end with  $L^2$ , there are pathologically many conjugate points. The main question we ask here is could the  $\dot{H}^{1/2}$  metric be in some sense critical? Could it lie along this spectrum in such a way that its geodesics admit infinite sequences of conjugate points that are dense but only at the blowup point along the geodesic? This is why, along with its similarities to the 3D Euler equations discussed previously that we are keenly interested in the possibility of conjugate cascades in the  $\dot{H}^{1/2}$  metric. A solution to this problem could provide critical insight into the geometric nature of blow-up for all these equations.

*Proof.* Formula (2.21) follows from writing down the relevant quantities in formula (1.52). They can be found in Misiołek-Preston [53]. We now show that for any initial function  $u_0 > 0$  with geodesic

$\eta$ , and between any two times  $a$  and  $b$ , we can find a function  $v$  such that  $I(J, J) < 0$ . Again, by lemma 1.2.1 this implies that there is a conjugate point along  $\eta$  between  $a$  and  $b$ . Let us make a  $v(x, t)$  of the form

$$v(x, t) = g(t) \cos(nx + t)$$

where  $g$  is a bump function satisfying  $g(a) = 0$  and  $g(b) = 0$ . Then

$$\begin{aligned} v_t^2 &= (g')^2 \cos^2(nx + t) - 2gg' \cos(nx + t) \sin(nx + t) + g^2 \sin^2(nx + t) \\ &\leq 2((g')^2 \cos^2(nx + t) + g^2 \sin^2(nx + t)) \leq 2((g')^2 + g^2) \end{aligned}$$

and

$$\begin{aligned} \dot{v}v_x - \dot{v}_x v &= (g' \cos(nx + t) - g \sin(nx + t))(-ng \sin(nx + t)) \\ &\quad - (-ng' \sin(nx + t) - ng \cos(nx + t))(g \cos(nx + t)) = ng^2 \end{aligned}$$

Then,

$$I(J, J) \leq \int_a^b \int_{S^1} 2\eta_x^2((g')^2 + g^2) - nu_0 g^2 dx dt$$

Letting  $n$  get arbitrarily large then forces  $I(J, J)$  to become arbitrarily negative.  $\square$

#### 2.4.2 The Search for an $\dot{H}^{1/2}$ Conjugate Cascade

**The Main Question:** Given a solution to the Wusnch equation as in Bauer-Kolev-Preston, does the trajectory pass through a conjugate cascade? That is, given the blow-up time  $T$ , is there a sequence of times  $t_n \rightarrow T$  such that  $\eta(t_n)$  is monoconjugate to  $\eta(t_{n+1})$ ?

From the vorticity conservation law we have that

$$\omega(t, \eta(t, x)) = \frac{\omega_0(x)}{\eta_x(\tau, x_0)^2}$$

Given Theorem 8, as is discussed in Bauer-Kolev-Preston [4], the existence of a conjugate cascade is equivalent to having a localized Beale-Kato-Majda criteria, which is currently unknown for this equation.

### 2.4.2.1 The Need for Refined Estimates

Throughout this document we will denote the blow-up point by  $x_0$ . It's known (again from Bauer-Kolev-Preston [4]) that as  $t \rightarrow T$ ,  $\eta_x(t, x_0) \rightarrow 0$ , but Theorem 8 doesn't help us (at least not immediately) as  $w_0(x_0) = 0$ . The hope is that if we select points  $x$  close to  $x_0$ , we'll get that equation 1.53 can be satisfied for pairs of times approaching  $T$ . Assume that  $u$  is odd about  $x_0$  so that  $u(t, x_0) = 0$ ,  $u_{xx}(t, x_0) = 0$ , and  $\eta(t, x_0) = x_0$ . Then choose  $x$  so that  $|x - x_0| < \epsilon$ . By differentiating the flow equation

$$\begin{aligned} \eta_{txx}(t, x_0) &= u_{xx}(t, \eta(t, x_0))(\eta_x)^2 + u_x(t, \eta(t, x_0))\eta_{xx} \\ &= u_x(t, \eta(t, x_0))\eta_{xx} \end{aligned}$$

we obtain that  $\eta_{txx}(t, x_0) = 0$  for all time (since  $\eta_{xx}$  is exponential in  $u_x$  but 0 at time 0). We will proceed according to the following heuristic argument. Expand

$$\eta_x(t, x) = \eta_x(t, x_0) + \epsilon^2 \eta_{xxx}(t, x_0) + O(\epsilon^3)$$

$$\omega_0(x) = \epsilon \omega'_0(x_0) + O(\epsilon^2).$$

Then the leading order terms from the numerator and denominator of 1.53 give

$$|\omega_0(x)| \int_a^b \frac{d\tau}{\eta_x(\tau, x)^2} \sim |\epsilon \omega'_0(x_0)| \int_a^b \frac{d\tau}{\eta_x(\tau, x_0)^2}. \quad (2.22)$$

Now we know from Bauer-Kolev-Preston [4] that  $\eta_x(t, x_0) \leq 1 - tu'_0(x_0)$ . So, we can write down

$$|\epsilon \omega'_0(x_0)| \int_a^b \frac{d\tau}{\eta_x(\tau, x_0)^2} \geq |\epsilon \omega'_0(x_0)| \int_a^b \frac{d\tau}{(1 - tu'_0(x_0))^2} \quad (2.23)$$

This could be the localized Beale-Kato-Majda criteria we've been looking for, but there are two problems. The first is that we need to rigorously justify the approximation 2.22, and the second (and more serious problem) is that we only know that the actual blow up time  $T$  satisfies  $T < |1/u'_0(x_0)|$ , which means that we're quite likely not giving the right side of 2.23 the time it needs to blow up, which coupled with the small size of  $\epsilon$  means that we don't actually know that this lower bound is good enough. What this means is that we need more refined estimates on either  $T$ , or  $\eta_x(t, x_0)$  or both.

#### 2.4.2.2 Estimating $F$

One of the main results of Bauer-Kolev-Preston [4] is that along  $x_0$ , the geodesic equation can be expressed as

$$\eta_{xtt}(t, x_0) = -F(t, x_0)\eta_x(t, x_0).$$

So in other words, controlling  $\eta_x$  and  $T$  amounts to gaining a better understanding of  $F$ . Now when we're on  $\mathbb{R}$  we have, from [72]:

$$F(t, x) = \|v^x(y)\|_{\dot{H}^{1/2}}, \quad (2.24)$$

where  $v^x(y) = \frac{u(y)-u(x)}{x-y}$ . The goal is to employ analysis techniques to provide estimates on  $F$ . Unfortunately, this is still an active field of research and we don't currently know enough to obtain the bounds we need. What follows are some more representations of  $F$  that may be of use.

#### 2.4.2.3 The Difference Quotient Approach

We have the following theorem from Bahouri-Chemin-Danchin [3]:

**Theorem 19** (Bahouri-Chemin-Danchin [3]). *Let  $s$  be a real number in the interval  $(0, 1)$  and  $u$  be in  $\dot{H}^s(\mathbb{R}^d)$ . Then,*

$$u \in L_{loc}^2(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dx dy < \infty.$$

Moreover, a constant  $C_s$  exists such that for any function  $u$  in  $\dot{H}^s(\mathbb{R}^d)$ , we have

$$\|u\|_{\dot{H}^s}^2 = C_s \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dx dy \quad (2.25)$$

Combining this representation of the  $\dot{H}^s$  norm with equation 2.24 we obtain a formula for  $F$  essentially as the integral of a double difference quotient:

**Theorem 20.**

$$F(t, x_0) = \int_{\mathbb{R}^2} \frac{(xu(x+y+x_0) - (x+y)u(x+x_0) + yu(x_0))^2}{x^2 y^2 (x+y)^2} dx dy \quad (2.26)$$

*Proof.* The proof follows immediately from the two aforementioned equations 2.24 and 2.25.  $\square$

In addition we have the following additional fact:

**Theorem 21** (Bahouri-Chemin-Danchin [3]). *The space  $L_{loc}^1(\mathbb{R}) \cap \dot{H}^{d/2}$  is included in  $BMO(\mathbb{R}^d)$ .*

Moreover, there exists a constant  $C$  such that

$$\|u\|_{BMO} \leq C \|u\|_{\dot{H}^{d/2}} \quad (2.27)$$

for all functions  $u \in L_{loc}^1(\mathbb{R}^d) \cap \dot{H}^{d/2}(\mathbb{R}^d)$ .

In other words the BMO norm of  $v^x$  forms a nontrivial lower bound for  $F$ .

## 2.5 Some Open Problems

In this section we put many of the open problems resulting from this work in one place.

- So far, all known examples have yielded positive curvature of the  $\dot{H}^{1/2}$  metric on  $\mathcal{D}(S^1)/S^1$  as in Chhay [7]. However, the explicit computation has proven to be difficult. It's currently believed that the existence of positive curvature may contribute significantly to non-Fredholmness and blow-up itself, so understanding this problem would be significant.

- In addition to the  $\dot{H}^{1/2}$  metric, Bauer-Kolev-Preston [4] studied the  $\mu H^{1/2}$  metric, which allows one to consider velocity fields with non-zero mean. What results of blow-up, curvature, etc., carry over into this context?
- Much remains unknown about the Teichmüller perspective of the  $\dot{H}^{1/2}$  metric. For example, in this dissertation we have performed numerical simulations mapping the Wunsch flow to the space of curves by employing conformal welding. More generally, can we develop a better sense of what the flow looks like in the universal Teichmüller curve? Perhaps the curve perspective is not the best. We might try the manifold of metrics, or moduli spaces of complex curves to find a natural setting. Teo [77] showed that integrating the square of the symplectic form in the Weil-Petersson ( $\dot{H}^{3/2}$ ) geometry over the  $PSL_2$  fibres gives the symplectic form in the Velling-Kirillov ( $\dot{H}^{1/2}$ ) geometry. Can one somehow use this to better connect the geometries of the two spaces? Perhaps one can capture how this affects critical features such as Fredholmness or blow-up.
- As geodesics for Wunsch approach their singular times, do they pass through conjugate cascades? If there is a useful picture of the Wunsch flow in terms of Teichmüller theory, what might a conjugate point or cascade look like there?
- Finally, the work of Escher-Wunsch [24] demonstrated that many of the equations in the generalized Constantin-Lax-Majda family cannot be realized as geodesic equations in the way that the Wunsch equation can. However they can be realized as non-metric Euler equations. The big question here is what knowledge from the Wunsch equation can we port to this other scenario?

## Chapter 3

### The SQG Equation as a Geodesic Equation

This section is taken from my paper Washabaugh [81]. In this paper, building off the work of Tao [76], I laid the foundations for the interpretation of the SQG equation (1.29) as an Euler-Arnold equation.

#### 3.1 Introduction

As discussed by Choi et al. [9], there is a large number of model equations of the full 3D Euler equations that have been investigated analytically. Some of these equations arise naturally as geodesic equations of right-invariant metrics of diffeomorphism and volume-preserving diffeomorphism (volumorphism) groups. For example, a special case of the generalized Constantin-Lax-Majda model first discussed by Okamoto et al. [58] is the Wunsch equation [83],

$$\omega_t + u\omega_x + 2\omega u_x = 0, \quad \omega = Hu_x, \quad (3.1)$$

which is the geodesic equation on the diffeomorphism group of the circle in the  $\dot{H}^{1/2}$  right-invariant metric. For equations arising in such a fashion, it is then natural to investigate their associated geometric properties in the manner initiated by V. Arnold [1]. In this paper, we demonstrate that the well known surface quasi-geostrophic (SQG) equation is the geodesic equation on the volumorphism group of a 2D manifold in the  $\dot{H}^{-1/2}$  inner product. The SQG equation on a Riemannian manifold

$M$  with metric  $\langle \cdot, \cdot \rangle$  is given by

$$\theta_t + \langle u, \nabla \theta \rangle = 0, \quad u = \mathcal{R}^\perp \theta, \quad (3.2)$$

where  $\mathcal{R}^\perp$  is the perpendicular Riesz transform. Many of the basic mathematical properties of this equation were first investigated by Constantin-Majda-Tabak [12]. Importantly, while this equation is known to have solutions for short time, the global in time existence problem is still open. It is believed by some (see e.g. Constantin-Majda-Tabak [12]) that the blow-up mechanism (should it exist) of this equation may have very similar properties to that of the full 3D Euler equations. As Bauer-Kolev-Preston [4] did for the Wunsch equation (3.1), in this paper we investigate some of the basic geometric properties of the SQG equation (3.2). We perform the necessary computations in a variety of domains in order to keep the paper as simple as possible. Looking forward, it will be necessary to firmly establish the theory of this equation in a single domain (as in Escher-Kolev [26] for positive fractional order Sobolev metrics on the diffeomorphism group of the circle).

The following is a list of the geometric properties associated to the SQG equation we explore:

- **Smoothness of the Riemannian exponential map**

The Riemannian exponential map on the volumorphism group in a Riemannian metric takes a velocity field (tangent vector) to the solution of the geodesic equation of the metric at time one. In our case, the geodesic equation is equivalent to SQG (3.2) and the geodesic evaluated at time one is a particle trajectory map. We may then ask whether or not this map is smooth. This question is partially answered by Constantin-Vicol-Wu [15] where the authors demonstrate analyticity of the particle trajectories. Here, we are also concerned with smooth dependence on the initial data. In this paper we demonstrate that the Lagrangian formulation of SQG has smooth dependence on the initial data in the case that the underlying manifold is  $\mathbb{R}^2$ . This suggests in general that the Riemannian exponential map will

be smooth for the  $\dot{H}^{-1/2}$  right invariant metric on the volumorphism group of any manifold.

- **Non-Fredholmness of the Riemannian exponential map**

Next, we show that the Riemannian exponential map on  $\mathcal{D}_\mu(\mathbb{S}^2)$  in the  $\dot{H}^{-1/2}$  inner product is not a Fredholm map in the sense of Smale [74]. Ebin et al. [20] showed that, for  $M$  a compact 2D Riemannian manifold without boundary, in the  $L^2$  metric on  $\mathcal{D}_\mu(M)$ , the exponential map is a nonlinear Fredholm map of index zero. It was also demonstrated that the exponential map is **not** Fredholm in the 3D situation. This points to a significant difference between 2D and 3D hydrodynamics. Fredholmness has been used to obtain results about the  $L^2$  geometry of the 2D volumorphism group, such as an infinite dimensional version of the Morse Index Theorem (see Misiołek and Preston [53]), and a version of the Morse-Littauer Theorem (see Misiołek [54]). In this paper, we solve the Jacobi equation along a simple rotational flow to demonstrate the existence of an epiconjugate point that is not monoconjugate (see Grossman [32]); thus the exponential map is non-Fredholm. Preston [65] showed that there is a concrete connection between blow up and the existence of conjugate points, thus our argument here provides evidence that the blow up behavior of 2D SQG is similar geometrically to that of 3D Euler.

- **The sectional curvature of the volumorphism group in the  $\dot{H}^{-1/2}$  metric and vanishing geodesic distance**

Finally, as was suggested by Arnold [1], the sectional curvature of the volumorphism group is helpful in predicting Lagrangian stability in fluid flows. While intuitively appealing, little is currently known about this relationship. One would like to be able to use the Rauch Comparison test; however to make use of this theorem, one must bound the sectional

curvature with either a strictly positive or strictly negative constant. For the  $L^2$  metric, various partial situations were investigated by Preston [62] in which it was demonstrated that the situation is quite complicated if one does not have these bounds. In this paper we demonstrate that  $\mathcal{D}_\mu(\mathbb{T}^2)$  (the volumorphism group of the flat torus) in the  $\dot{H}^{-1/2}$  metric exhibits arbitrarily large curvature of both signs. As was first conjectured by Michor and Mumford [50], we conjecture that the unbounded curvature implies that the geodesic distance on this space vanishes.

### 3.2 The SQG equation as a geodesic equation

Tao [76] demonstrated that solutions to the SQG equation are the critical points of a functional obtained from the inertia operator  $A = (-\Delta)^{-1/2}$ . Assume that  $M$  is a 2D Riemannian manifold, possibly with boundary. It is known from Arnold [1] that, in the case that  $M$  is compact  $\mathcal{D}_{\mu,ex}(M)$ , the group of exact volumorphisms, can be thought of as an infinite-dimensional Lie-group. In this section, we demonstrate that the SQG equation is the geodesic equation on  $\mathcal{D}_{\mu,ex}(M)$  in the  $\dot{H}^{-1/2}$  metric, obtained from the inertia operator  $A$  (we will be working on this space formally in the case that  $M$  is not compact). However, in comparison with Bauer-Kolev-Preston [4] we consider the  $\dot{H}^{1/2}$  metric on  $C^\infty(M)$ . In other words, for  $\phi, \psi \in C^\infty(M)$  such that  $\phi|_{\partial M}, \psi|_{\partial M} = 0$ ,

$$\langle\langle \phi, \psi \rangle\rangle_{\dot{H}^{1/2}} = \int_M ((-\Delta)^{1/2}\phi)\psi d\mu. \quad (3.3)$$

We can make this into a metric on  $T_{id}\mathcal{D}_{\mu,ex}(M)$  by letting  $u = \nabla^\perp\phi$  and  $v = \nabla^\perp\psi$  which gives us:

$$\langle\langle u, v \rangle\rangle_{\dot{H}^{-1/2}} = \int \langle (-\Delta)^{-1/2}u, v \rangle d\mu. \quad (3.4)$$

In other words, a possible analogy to the 1D  $\dot{H}^{1/2}$  metric is the  $\dot{H}^{-1/2}$  metric on  $\mathcal{D}_{\mu,ex}(M)$ . Using push-forward by right translation we then obtain a right invariant metric on all of  $\mathcal{D}_{\mu,ex}(M)$ . Now,

on  $C^\infty(M)$  the Euler-Arnold equation is given by

$$\psi_t = -\text{ad}_\psi^* \psi, \quad (3.5)$$

where  $\text{ad}_\psi^* : \mathfrak{g} \rightarrow \mathfrak{g}$  is given by

$$\langle \langle \text{ad}_\psi^* \phi, \nu \rangle \rangle_{\dot{H}^{1/2}} = \langle \langle \phi, \text{ad}_\psi \nu \rangle \rangle_{\dot{H}^{1/2}},$$

and  $\text{ad}_\psi \nu = -\{\psi, \nu\} = \{\nu, \psi\}$  is the negative of the Poisson bracket, which for our purposes will be given by

$$\{\nu, \psi\} = d\nu(\nabla^\perp \psi) = \langle \nabla \nu, \nabla^\perp \psi \rangle.$$

**Remark.** Our  $\text{ad}^*$  operator here defined on the Lie algebra  $\mathfrak{g}$  is the same as that used in [53], which is slightly different from the usual  $\text{ad}^*$  operator defined on the dual Lie algebra  $\mathfrak{g}^*$ .

**Theorem 22.** The SQG equation is the geodesic equation on  $D_{\mu,ex}(M)$  and on  $D_\mu(M)$  in the  $\dot{H}^{-1/2}$  metric.

*Proof.* For  $\mathcal{D}_{\mu,ex}(M)$ , we compute  $\text{ad}^*$  on  $C^\infty(M)$ . For  $\psi, \phi, \nu \in C^\infty(M)$  such that  $\psi|_{\partial M}, \phi|_{\partial M}, \nu|_{\partial M} = 0$ ,

$$\begin{aligned} \langle \langle \text{ad}_\psi^* \phi, \nu \rangle \rangle &= \langle \langle \phi, \text{ad}_\psi \nu \rangle \rangle = \langle \langle \phi, \{\nu, \psi\} \rangle \rangle \\ &= \int_M A(\phi) \{\nu, \psi\} d\mu = \int_M A(\phi) \langle \nabla \nu, \nabla^\perp \psi \rangle d\mu \\ &= - \int_M \nu \text{div}(A(\phi) \nabla^\perp \psi) d\mu + \int_{\partial M} \nu \langle A(\phi) \nabla^\perp \psi, n \rangle d\tilde{\mu} = - \int_M \nu \langle \nabla A(\phi), \nabla^\perp \psi \rangle d\mu, \end{aligned}$$

where  $n$  is the unit normal to the boundary and  $d\tilde{\mu}$  is the boundary measure. Note then, that the boundary term vanishes. Thus

$$\text{ad}_\psi^* \phi = A^{-1} \left( - \left\langle \nabla A(\phi), \nabla^\perp \psi \right\rangle \right),$$

and the geodesic equation (3.5) becomes

$$A\psi_t = - \left\langle \nabla A\psi, \nabla^\perp \psi \right\rangle.$$

Letting  $A\psi = \theta$  and  $u = \nabla^\perp \psi$  we obtain the SQG equation (3.2). The case for  $\mathcal{D}_\mu(M)$  follows by computing  $\text{ad}^*$  for vector fields in the  $\dot{H}^{-1/2}$  metric and then applying  $\nabla \times A$  to both sides of the equation.  $\square$

**Remark.** *Note that if  $M$  admits harmonic vector fields, then this inner product is degenerate on  $\mathcal{D}_\mu(M)$ . Thus in these situations we are really considering this as a geodesic equation on a homogenous space. One needs to verify that the inertia operator is invariant with respect to  $Ad$  as is done in Khesin and Misiołek [37]. A short computation shows that this holds.*

### 3.3 Smoothness of the Riemannian Exponential Map

The Riemannian exponential map on a Riemannian manifold  $\mathbf{N}$  at a point  $(p, v) \in T\mathbf{N}$ , the tangent bundle of  $\mathbf{N}$  is given by

$$\exp : T\mathbf{N} \rightarrow \mathbf{N}$$

$$\exp_p(v) = \gamma_v(1), \tag{3.6}$$

where  $v \in T_p\mathbf{N}$  and  $\gamma_v(1)$  is the geodesic through  $p$  with initial velocity  $v$  evaluated at time 1. As in Constantin-Vicol-Wu [15], we may write SQG (3.2) as an ODE on a Banach manifold  $\mathbf{M}$  (to be defined below), which will correspond to a sub-manifold of  $T\mathbf{N}$ . The ODE will look like

$$\frac{dX}{dt} = F(X, \theta_0). \tag{3.7}$$

Then  $p$  and  $v$  will correspond to  $X$  and  $\theta_0$  respectively. Smoothness of the Riemannian exponential map is then equivalent to the above equation having smooth (in time) solutions that vary smoothly with respect to the initial data. There are results establishing smoothness of exponential maps in general Sobolev metrics. For example, Escher-Kolev [26] did this for Sobolev metrics of order  $s \geq \frac{1}{2}$  on the diffeomorphism group of the circle. However, our Sobolev metric is of **negative** index, thus no known results apply. Constantin-Vicol-Wu [15] demonstrated that the individual particle paths are analytic as functions of time. They proved the following theorem that we cite here for convenience:

**Theorem 23** (Constantin-Vicol-Wu [15]). *Consider initial data  $\theta_0 \in C^{1,\gamma} \cap W^{1,1}$ , and let  $\theta$  be the unique maximal solution to (3.2), with  $\theta \in L_{loc}^\infty([0, T_*]; C^{1,\gamma} \cap W^{1,1})$ . Given any  $t \in [0, T_*)$ , there exists  $T \in (0, T_* - t)$ , with  $T = T(\|\nabla u\|_{L^\infty(t, (t+T_*)/2; L^\infty)})$ , and  $R > 0$  with  $R = R(t, \|\theta_0\|_{C^{1,\gamma} \cap W^{1,1}}, \gamma)$ , such that*

$$\|\partial_t^n (X - id)\|_{L^\infty(t, t+T; C^{1,\gamma})} \leq Cn!R^{-n} \quad (3.8)$$

holds for any  $n \geq 0$ . Here  $C$  is a universal constant, and the norm  $\|X - id\|_{C^{1,\gamma}}$  is defined in equation (3.9). In particular, the Lagrangian trajectory  $X$  is a real analytic function of time, with radius of analyticity  $R$ .

Our purpose here is to demonstrate that the Riemannian exponential map is smooth, which is equivalent to demonstrating smooth dependence on the initial data ( $X$  and  $\theta_0$ ) in (3.7). We will do so not on  $T\mathbf{N}$  but instead on a closely related Banach affine space, denoted by  $\mathbf{M}$  to be defined below. We will show how the argument from Constantin-Vicol-Wu [15] can be extended to obtain smoothness of  $F$  so that the following theorem from Lang (Chapter 4, Theorem 1.11 in [43]) can be applied:

**Theorem 24** (Lang [43]). *Let  $J$  be an open interval in  $\mathbb{R}$  containing 0 and  $U$  open in the Banach space  $\mathbf{E}$ . Let*

$$f : J \times U \rightarrow \mathbf{E}$$

be a  $C^p$  map with  $p \geq 1$ , and let  $x_0 \in U$ . There exists a unique local flow for  $f$  at  $x_0$ . We can select an open subinterval  $J_0$  of  $J$  containing 0 and an open subset  $U_0$  of  $U$  containing  $x_0$  such that the unique local flow

$$\eta : J_0 \times U_0 \rightarrow U,$$

is of class  $C^p$ , and such that  $D_2\eta$  satisfies the differential equation

$$D_1D_2\eta(t, x) = D_2f(t, \eta(t, x))D_2\eta(t, x)$$

on  $J_0 \times U_0$  with initial condition  $D_2\eta(0, x) = id$ .

Note that in our case,  $f = F$  will be autonomous, so we will only need smoothness in  $U$ . This in turn demonstrates smoothness of the Riemannian exponential map in this situation. In fact, most of the argument has been done in Constantin-Vicol-Wu [15]. Here we are extending their argument to obtain smooth dependence of the initial data and hence smoothness of the Riemannian exponential map.

**Remark.** As Constantin-Vicol-Wu [15] demonstrated analyticity of the Lagrangian trajectories, one can quite likely extend these arguments to obtain analyticity of the Riemannian exponential map as was done by Shnirelman [71] for the  $L^2$  metric on  $\mathcal{D}_\mu^s T^3$ , the group of order  $s$  Sobolev class volumorphisms for  $s > 5/2$ .

### 3.3.0.4 The Domain M

Following the strategy of Chapter 4 of Majda-Bertozzi [48] we enlarge the space of volume preserving maps to allow for maps with some compressibility. This allows us to apply Theorem 24 directly as we can deal with an open subset of a Banach space rather than a submanifold.

Constantin-Vicol-Wu [15] analyze SQG explicitly on the volume preserving case, however, as they mention, their argument extends to the compressible case. We let

$$\mathbf{N} = C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2),$$

for

$$C_b^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2) = \{Y : \mathbb{R}^2 \rightarrow \mathbb{R}^2 : \|Y\|_{1,\gamma} < \infty\},$$

$$C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2) = \{id + C_b^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2)\},$$

where

$$\|Y\|_{1,\gamma} = \|Y\|_{L^\infty} + \|\nabla Y\|_{L^\infty} + [\nabla Y]_{C^\gamma}. \quad (3.9)$$

Here, the  $L^\infty$  norm is taken as the largest absolute value of an entry of the corresponding vector or matrix. Note that  $C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2)$  is an affine Banach space. Then we may identify

$$TN = C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2) \times C_b^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2).$$

Points in  $TN$  are of the form  $(X, u)$ . We define the domain on which we'll be solving SQG to be

$$\mathbf{M} = C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2) \times (C^{1,\gamma}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)),$$

where  $C^{1,\gamma}(\mathbb{R}^2)$  is the space of Hölder continuous functions on  $\mathbb{R}^2$  and  $W^{1,1}(\mathbb{R}^2)$  is the corresponding Sobolev space of functions on  $\mathbb{R}^2$ . We note then that the perpendicular Riesz transform,

$$\mathcal{R}^\perp : C^{1,\gamma}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2) \rightarrow C_b^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2),$$

$$\mathcal{R}^\perp : \theta \mapsto u,$$

gives a correspondence between  $\mathbf{M}$  and a subset of  $T\mathbf{N}$ . We must also select the open set  $\mathcal{U} \subset \mathbf{M}$  on which we'll define  $F$ , as in the theorem from Lang. As discussed above, our problem is that we would like to focus only on  $X$  such that  $\det \nabla_a X(a) = 1$ , but this does not yield us an open subset of  $\mathbf{M}$ , thus as in chapter 4 of Majda-Bertozzi [48], we enlarge our domain to include some compressibility. We define

$$\mathcal{O} = \{X = id + Y \in C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2) : \frac{9}{10} < \inf_{a \in \mathbb{R}^2} \det \nabla_a X(a), \|Y\|_{1,\gamma} < c\},$$

$$\mathcal{U} = \mathcal{O} \times (C^{1,\gamma}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)),$$

where

$$c = \frac{7}{20}.$$

That  $\mathcal{O}$  and hence  $\mathcal{U}$  are open in their respective spaces follows from continuity of  $\inf_{a \in \mathbb{R}^2} \det$  and  $\|\cdot\|_{1,\gamma}$ . We will also require more properties of  $\mathcal{O}$  that are used by Constantin [15]. The main fact they use to obtain analyticity of the particle trajectories is the following chord-arc condition, which is satisfied by solutions of SQG with Lipschitz velocity field  $u$ . That is, there is a constant  $\lambda$  such that

$$\lambda^{-1} \leq \frac{|a-b|}{|X(a,t) - X(b,t)|} \leq \lambda. \quad (3.10)$$

We claim that there exists such a constant  $\lambda$  for  $X \in \mathcal{O}$ . First we need the following result:

**Lemma 3.3.1.** *Suppose that  $X = id + Y \in \mathcal{O}$ , then  $X$  is a homeomorphism of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$ .*

*Proof.* Since  $X \in \mathcal{O}$ ,  $\|\nabla Y\|_{L^\infty} < c$ . Hence the largest an entry of  $\nabla X$  can be in magnitude is  $c+1$ .

Writing out the inverse of  $\nabla X$  explicitly, combined with the fact that

$$\frac{9}{10} < \inf_{a \in \mathbb{R}^2} \det \nabla_a X(a)$$

yields:

$$\|\nabla X^{-1}\|_{L^\infty} < \frac{3}{2}.$$

As is similarly discussed in Majda-Bertozzi [48], a result of Hadamard ([6], pg. 222) demonstrates that if  $X \in \mathcal{O}$  and there exists a constant  $d$  such that

$$\|\nabla X^{-1}\|_{L^\infty} \leq d,$$

then  $X$  is a homeomorphism of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$ . □

In order to make use of the estimates of Constantin-Vicol-Wu [15], it is necessary that  $\lambda \in (1, \frac{3}{2}]$  in (3.10).

**Lemma 3.3.2.** *If  $X \in \mathcal{O}$ , then  $X$  satisfies the chord-arc condition (3.10) for  $\lambda = \frac{3}{2}$ .*

*Proof.* Since  $X \in C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2)$ , given  $a, b \in \mathbb{R}^2$ ,

$$|X(a) - X(b)| \leq |\nabla X|_{L^\infty} |a - b|.$$

Thus

$$\frac{1}{|\nabla X|_{L^\infty}} \leq \frac{|a - b|}{|X(a) - X(b)|}. \quad (3.11)$$

Hence,

$$|X^{-1}(\alpha) - X^{-1}(\beta)| \leq \|\nabla X^{-1}\|_{L^\infty} |\alpha - \beta| < \frac{3}{2} |\alpha - \beta|,$$

where we have used the bound on  $\|\nabla X^{-1}\|_{L^\infty}$  obtained in the proof of lemma 3.3.1. Choosing  $\alpha = X(a)$  and  $\beta = X(b)$  yields:

$$\frac{|a - b|}{|X(a) - X(b)|} < \frac{3}{2}.$$

This combined with (3.11) gives us the claim.

□

For the SQG equation, we recover the velocity field from the vorticity by

$$u(x) = \mathcal{R}^\perp \theta(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi|x-y|^3} \theta(y) dy = \int_{\mathbb{R}^2} K(x-y) \theta(y) dy,$$

where all integrals are considered in the principal value sense. The SQG equation itself says that

$$\theta(X(b, t), t) = \theta_0(b).$$

Hence from the flow equation we obtain

$$\frac{dX}{dt}(a, t) = \int_{\mathbb{R}^2} K(X(a, t) - y) \theta(y, t) dy. \quad (3.12)$$

Then the precise system of ODEs we will be studying is given by

$$\frac{dX}{dt} = F(X, \theta_0),$$

where

$$\begin{aligned} F(X, \theta_0)(a) &= \int_{\mathbb{R}^2} \frac{(X(a) - X(b))^\perp}{2\pi|X(a) - X(b)|^3} \theta_0(b) J_X(b) db \\ &= \int_{\mathbb{R}^2} K(X(a) - X(b)) \theta_0(b) J_X(b) db, \end{aligned} \quad (3.13)$$

is the Riesz transform of  $\theta_0$  when  $X = \text{id}$  and  $J_X(b) = \det \nabla_b X(b)$ . We also wish to obtain  $\nabla_a F(X, \theta_0)(a)$ . This follows in essentially the same manner as what is done by Constantin et al. [15],

$$\nabla_a F(X, \theta_0) = \nabla_a X(a, t) \int_{\mathbb{R}^2} K(X(a, t) - X(b, t)) (\nabla_b^\perp X^\perp)(b, t) (\nabla_b \theta_0)(b) J_X(b) db. \quad (3.14)$$

The fact that  $F$  is well defined is nontrivial, but follows from the smoothness argument. We will also need the following estimates on compositions of functions in  $C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2)$ .

**Lemma 3.3.3.** *Let  $X \in \mathcal{O}$ ,  $Z \in C^{1,\gamma}(\mathbb{R}^2, \mathbb{R}^2)$ . We have*

$$\|Z \circ X - id\|_{1,\gamma} \leq C_1(1 + \|Z - id\|_{1,\gamma}), \quad (3.15)$$

where  $C_1$  is determined entirely by  $\mathcal{O}$ .

*Proof.* By definition,

$$\|Z \circ X - id\|_{1,\gamma} = \|Z \circ X - id\|_\infty + \|\nabla(Z \circ X) - I\|_\infty + |\nabla(Z \circ X)|_\gamma,$$

where  $I$  is the identity matrix. There exists  $W \in C^{1,\gamma}(\mathbb{R}^2) \times C^{1,\gamma}(\mathbb{R}^2)$  such that  $Z = id + W$ . Hence for the first term above,

$$\|Z \circ X - id\|_\infty = \|X + W \circ X - id\|_\infty \leq c + \|Z - id\|_\infty, \quad (3.16)$$

where we recall that  $c = \frac{7}{20}$ . For the second term,

$$\|\nabla_a(Z \circ X)(a) - I\|_\infty = \|\nabla_a(X + W \circ X)(a) - I\|_\infty \leq c + \|\nabla_a(W \circ X)(a)\|_\infty \leq C(1 + \|\nabla Z - I\|_\infty) \quad (3.17)$$

for some constant  $C$ . Similarly, for the third term,

$$|\nabla_a(Z \circ X)(a)|_\gamma \leq C(1 + |\nabla_a Z(a)|_\gamma). \quad (3.18)$$

Combining (3.16), (3.17), and (3.18) gives us the claim.

□

We will also need a sense of how  $F$  behaves under composition of functions.

**Lemma 3.3.4.** *Let  $Y \in \mathcal{O}$ . Then*

$$F(X, \theta_0) \circ Y = F(X \circ Y, \theta_0 \circ Y).$$

*Proof.*

$$F(X, \theta_0) \circ Y = \int_{\mathbb{R}^2} \frac{(X(Y(a)) - X(b))^\perp}{2\pi|X(Y(a)) - X(b)|^3} \theta_0(b) J_X(b) db.$$

Let  $b = Y(s)$ , for  $b, s \in \mathbb{R}^2$ . Then

$$= \int_{\mathbb{R}^2} \frac{(X(Y(a)) - X(Y(s)))^\perp}{2\pi|X(Y(a)) - X(Y(s))|^3} \theta_0(Y(s)) J_X(Y(s)) J_Y(s) ds = F(X \circ Y, \theta_0 \circ Y).$$

□

### 3.3.1 Smoothness of the ODE

Since  $\mathcal{U}$  is an open subset of the affine space  $\mathbf{M}$ , we can demonstrate smoothness of  $F$  by showing that the operator norms of its partial derivatives,  $d_X^n(F)$  and  $d_{\theta_0}^n(F)$ , are bounded in some uniform way on  $\mathcal{U}$ . As discussed above, we show that  $F$  is smooth by adapting the argument made by Constantin et al. [15]. The theorem of Lang can be used after the following theorem:

**Theorem 25.**  *$F$  is infinitely Fréchet differentiable on  $\mathcal{U}$ .*

*Proof.* The idea is the following, we wish to obtain a bound on  $\|d_X^n F\|_{\mathcal{L}^n}$  (where  $\mathcal{L}^n$  is the corresponding space of multilinear maps):

$$\begin{aligned} \|d_X^n F(X_1, \dots, X_n)\|_{1,\gamma} &\leq \|d_X^n F\|_{\mathcal{L}^n} \cdot \|X_1\|_{1,\gamma} \cdots \|X_n\|_{1,\gamma} \\ &= \|d_X^n F\|_{\mathcal{L}^n}, \end{aligned}$$

for all  $X_i \in \partial B_1(0) \subset \mathbf{E}_1$  where  $\|d_X^n F\|_{\mathcal{L}^n}$  is independent of  $X$ . Now, if  $X(t)$  is a solution to SQG with initial condition  $\theta_0$  and with  $J_X = 1$ , Constantin et al. [15] estimates (for our purposes) for

$n \geq 0$ :

$$\|\partial_t^{n+1}|_{t=0} X(t)\|_{1,\gamma} = \|\partial_t^n|_{t=0} F(X(t), \theta_0)\|_{1,\gamma} \leq Cn!R^{-n}, \quad (3.19)$$

where  $R = R(\|\theta_0\|_{C^{1,\gamma} \cap W^{1,1}}, \gamma, \lambda) > 0$  and  $C$  is another constant. This gives us that the incompressible particle trajectories are analytic in time. The first point is that, as Constantin et al. [15] mentions, this argument can be extended for  $J_X \neq 1$ . The terms then involve a Jacobian and its time derivatives, but these are bounded, hence the same estimates go through but with modified constants. We now show that this bound also gives us that  $F$  is smooth in its  $X$  component, i.e. it provides the desired bound on  $\|d_X^n F\|_{\mathcal{L}^n}$ . In the case  $n = 0$ , bound (3.19) gives us that  $F$  is well defined at the identity, i.e.

$$\|F(id, \theta_0)\|_{1,\gamma} \leq CR^{-1}.$$

Away from the identity, we make use of lemmas 3.3.3 and 3.3.4 to obtain the desired bound. We will now proceed by induction. The idea is the following: using the multivariate Faá di Bruno formula we can expand  $\partial_t^n F(X(t), \theta_0)|_{t=0}$ , note in particular that the last term is  $d_X^n F(X_1, \dots, X_1)$  where  $X_1 = \partial_t|_{t=0} X(t)$ . Assuming that the bound holds in the case  $n - 1$ , we can subtract out bounded lower order terms from  $\partial_t^n F(X(t), \theta_0)|_{t=0}$  to obtain that  $d_X^n F(X_1, \dots, X_1)$  is bounded. One can then obtain a bound on the full operator  $d_X^n F(X_1, \dots, X_n)$  by polarization. This gives us smoothness at  $X = id$ . We will then use lemmas 3.3.3 and 3.3.4 to obtain smoothness for any  $Y \in \mathcal{O}$ . Here we do this explicitly for the case  $n = 2$ . Suppose that  $X(t)$  is a smooth curve in  $\mathcal{O}$  such that  $X(0) = X$  and  $\partial_t|_{t=0} X(t) = X_1$ , with  $\|X_1\|_{1,\gamma} = 1$ , and such that  $X$  is a solution to SQG. We have,

$$\begin{aligned} \partial_t^3|_{t=0} X(t) &= \partial_t^2|_{t=0} F(X(t), \theta_0) \\ &= (d^2 F)_X(X_1, X_1) + (dF)_X(\tilde{X}_2), \end{aligned}$$

where  $\tilde{X}_j = \partial_t^j \Big|_{t=0} X(t)$ , hence we may write

$$(d^2F)_X(X_1, X_1) = \partial_t^3 \Big|_{t=0} X(t) - (dF)_X(\tilde{X}_2).$$

If  $X = id$ , then by (3.19) and the inductive hypothesis we have that

$$\|(d^2F)_X(X_1, X_1)\|_{1,\gamma} \leq C(R^{-1} + 2R^{-2}).$$

Now, to obtain  $(d^2F)_X(X_1, X_2)$  for any other  $X_2$  we use polarization to obtain that

$$\begin{aligned} \|(d^2F)_X(X_1, X_2)\|_{1,\gamma} &= \frac{1}{2} \|(d^2F)_X(X_1 + X_2, X_1 + X_2) - dF_X(X_1, X_1) - dF_X(X_2, X_2)\|_{1,\gamma} \\ &\leq 2C(R^{-1} + 2R^{-2}). \end{aligned}$$

Now, if  $Z \in \mathcal{O}$ , one can verify, in a manner similar to lemma 3.3.4, that

$$(\partial_t^2 \Big|_{t=0} F(X(t), \theta_0)) \circ Z = (\partial_t^2 \Big|_{t=0} F(X(t), \theta_0) \circ Z).$$

This then gives us that:

$$(d^2F)_{X \circ Z}(X_1 \circ Z, X_2 \circ Z) = d^2F_X(X_1, X_2) \circ Z.$$

By lemma 3.3.3 we obtain the desired bound. Finally, we note that if  $\|\theta_0\|_{C^{1,\gamma} \cap W^{1,1}} = 1$ , then we have a bound  $F(X, \theta_0) \leq C$ . Since  $F$  is linear in  $\theta_0$ , this gives us that  $F$  is a bounded linear operator in  $\theta_0$ . Hence  $F$  is smooth in  $\theta_0$  and the Riemannian exponential map is smooth by Lang's Theorem 24.

□

### 3.4 Non-Fredholmness of the Riemannian Exponential Map

Preston [62] and Rouchon [69] demonstrated that just as the geodesic equation on the volume morphism group in the  $L^2$  metric splits in the Lie algebra, so does the Jacobi equation. Here, we

will make heavy use of Proposition 1.2.1, which gives us this fact. We find some explicit solutions of this equation which in turn give us non-Fredholmness. Let  $\mathbb{S}^2$  denote the standard two-sphere. In this section we demonstrate the following:

**Theorem 26.** *The Riemannian exponential map on  $\mathcal{D}_\mu(\mathbb{S}^2)$  in the  $\dot{H}^{-1/2}$  inner product is non-Fredholm.*

*Proof.* We will let  $u = \nabla^\perp f$ ,  $Y = \nabla^\perp g$ , and  $Z = \nabla^\perp h$ . Note then that (1.49) and (1.50) give us:

$$g_t + \{f, g\} = h, \quad (3.20)$$

$$\psi_t + \frac{1}{\sin \phi} (f_\phi \psi_r - f_r \psi_\phi) + \frac{1}{\sin \phi} (h_\phi \theta_r - h_r \theta_\phi) = 0, \quad (3.21)$$

respectively, where  $\psi = \sqrt{-\Delta}(h)$ . Here  $f$  (and hence  $\theta$ ) will be determined by a solution to SQG. We must impose the condition that  $g(0) = 0$ , so that we have a proper Jacobi field. Let  $f = -\cos \phi$ . Then, since  $\Delta f = 2 \cos \phi$  we have that

$$A(\cos(\phi)) = \sqrt{-\Delta}(\cos \phi) = \sqrt{2} \cos \phi.$$

We note that  $u$  and  $\theta = \nabla \times A(u)$  solve the SQG equation in spherical coordinates. Then, we let

$$h = \sum h_{nm}(t) \xi_{nm}(\phi) e^{imr}$$

where  $\xi_{nm}(\phi) e^{imr}$  is an eigenfunction of  $\Delta$ :

$$\Delta \xi_{nm}(\phi) e^{imr} = -\lambda_n^2 \xi_{nm}(\phi) e^{imr},$$

and  $\lambda_n = \sqrt{n(n+1)}$  with  $-n \leq m \leq n$ . The solution to (3.21) is

$$h_{nm}(t) = C_{nm} \cdot \text{Exp}\left[\frac{i(\sqrt{2} - \lambda_n)}{\lambda_n} mt\right].$$

Solving (3.20) for  $g$  we obtain

$$g_{nm}(t) = \frac{-iC_{nm}}{(1+a_n)m} e^{-int} \left( e^{i(1+a_n)mt} - 1 \right),$$

where

$$a_n = \frac{(\sqrt{2} - \lambda_n)}{\lambda_n}.$$

$g_{nm}(t)$  will be zero at

$$t_{nm} = \frac{2\pi}{(a_n + 1)m} = \frac{2\pi\sqrt{n(n+1)}}{\sqrt{2}m}.$$

This gives us that

$$\lim_{n \rightarrow \infty} t_{nn} = \frac{2\pi}{\sqrt{2}}.$$

Thus we have a clustering of conjugate points at  $t = \frac{2\pi}{\sqrt{2}}$ . So this is a point that is epiconjugate, but not monoconjugate, hence the map is not Fredholm.

□

### 3.4.1 The Sign and Magnitude of the Sectional Curvature

For a Lie group  $G$  with right invariant metric  $\langle \cdot, \cdot \rangle$  the non-normalized sectional curvature at the identity in directions  $u$  and  $v$  is given (as in Arnold [1]) by

$$\begin{aligned} \bar{K}(u, v) &= \langle \langle R(u, v)v, u \rangle \rangle = \frac{1}{4} \|\text{ad}_v^* u + \text{ad}_u^* v\|^2 - \langle \langle \text{ad}_u^* u, \text{ad}_v^* v \rangle \rangle \\ &\quad - \frac{3}{4} \|\text{ad}_u v\|^2 + \frac{1}{2} \langle \langle \text{ad}_u v, \text{ad}_v^* u - \text{ad}_u^* v \rangle \rangle. \end{aligned}$$

The normalized sectional curvature is given by

$$K(u, v) = \frac{\bar{K}(u, v)}{\|u\|^2 \|v\|^2 - \langle \langle u, v \rangle \rangle^2}. \quad (3.22)$$

Khesin et al. [38] computed  $\overline{K}$  explicitly for homogeneous Sobolev metrics on  $\mathcal{D}_{\mu,ex}(\mathbb{T}^2)$  for vector fields of the form  $u = \nabla^\perp \cos(jx + ky)$  and  $v = \nabla^\perp \cos(lx + my)$ . Here, one may consider the Lie algebra to be  $C^\infty(\mathbb{T}^2)$ . Then, given a metric  $\langle\langle, \rangle\rangle$  on  $\mathcal{D}_{\mu,ex}(\mathbb{T}^2)$  we obtain an inner product on  $C^\infty(\mathbb{T}^2)$  given by

$$\langle\langle u, v \rangle\rangle = \langle\langle \nabla^\perp f, \nabla^\perp g \rangle\rangle = \int_{\mathbb{T}^2} f \Lambda g d\mu = \langle\langle f, g \rangle\rangle,$$

for some positive definite, symmetric operator  $\Lambda$ . For our purposes we will have  $F(p) = \sqrt{j^2 + k^2}$  where  $F$  is the symbol of  $\Lambda$  and  $p = (j, k)$ .

**Proposition 3.4.1** (Khesin et al. [38]). *Suppose  $f(x, y) = \cos(jx + ky)$  and  $g(x, y) = \cos(lx + my)$  where  $j, k, l, m$  are integer multiples of  $2\pi$ . Set  $p = (j, k)$  and  $q = (l, m)$ , and let  $u = \nabla^\perp f$  and  $v = \nabla^\perp g$ . Then the non-normalized sectional curvature is given by*

$$\begin{aligned} \overline{K}(u, v) = \frac{|p \wedge q|^2}{8} & \left\{ \frac{1}{4} (F(p) - F(q))^2 \left( \frac{1}{F(p+q)} + \frac{1}{F(p-q)} \right) \right. \\ & \left. - \frac{3}{4} (F(p+q) + F(p-q)) + F(p) + F(q) \right\}, \end{aligned} \quad (3.23)$$

where  $p \wedge q = jm - kl$ .

Normalizing the above formula to obtain the usual sectional curvature we have the following:

**Theorem 27.** *The sectional curvature of  $\mathcal{D}_{\mu,ex}(\mathbb{T}^2)$  in the  $\dot{H}^{-1/2}$  metric is unbounded of both signs.*

*Proof.* Let  $n \in \mathbb{N}$ . First we choose  $j = m = 2\pi n$  and  $k = l = 0$ . Then using (3.22) and (3.23) we obtain

$$K(\cos(2n\pi x), \cos(2n\pi y)) \approx -15.0 n^3,$$

which demonstrates that the sectional curvature can be made to be arbitrarily negative for arbitrarily large  $n$ .

Next we choose  $j = m = l = n$  and  $k = 0$ . Then

$$K(\cos(2n\pi x), \cos(2n\pi x + 2n\pi y)) \approx 4.3 n^3,$$

which demonstrates that the curvature can be made to be arbitrarily positive for large  $n$ .  $\square$

### 3.5 More Open Problems

As in the previous chapter, we once more list some of the many open problems that result from this work. While the SQG equation is given by the  $\dot{H}^{-1/2}$  metric on vector fields, it's given by the  $\dot{H}^{1/2}$  metric on stream functions in the same way that the Wunsch equation is. Many of the currently known geometric properties of SQG are very similar to that of Wunsch (non-Fredholmness, arbitrarily large curvature). Given that SQG was originally created as a higher dimensional analogue to the generalized Constantin-Lax-Majda equation from which Wunsch descends, this is a tantalizing connection which motivates many of the directions in which future work on this should go.

- So far we have found a single clustering of conjugate points to demonstrate non-Fredholmness of the Riemannian exponential map. The next natural direction is to develop more global criteria for the existence of such points such as Theorem 8 from Bauer-Kolev-Preston [4].
- Given such a criteria for the existence of conjugate points, can we find conjugate cascades? We should expect this problem to be exceedingly difficult since blow-up is quite closely related to conjugate cascades and blow-up remains a difficult open problem. However, we may be able to develop criteria as in Preston [65] such as if one approaches a potential blow-up point the geodesic passes through a conjugate point or some condition on the stretching matrix is satisfied. This is a very interesting and approachable problem to see what happens from the SQG perspective.
- While currently beyond our abilities, one cannot help but wonder at the possible geometric connections of SQG in a way similar to Wunsch for Teichmüller theory. There are many possibilities here. For example Preston [68] showed that the perpendicular Riesz transform

satisfies a Cauchy-Riemann-Feuter system for quaternionic-differentiable functions in a way similar to the Hilbert transform for complex-differentiable functions. Unfortunately the problems rapidly get very difficult when we go in this direction. Perhaps instead there exists some sort of Kähler structure in this metric. Alternatively, maybe one can develop a sensible picture from a manifold of mappings perspective as Tromba [79] did for the Weil-Petersson equation.

## Chapter 4

### The Geometry of Axisymmetric Ideal Fluid Flows with Swirl

This section was originally inspired by the computation done in Preston [63], where the curvature of  $\mathcal{D}_\mu(M)$  was computed for rotational flows around a variety of 2D manifolds. In that paper, the curvature was generally found to be negative. The idea here was to compute the curvature around similar flows in 3D. In particular, Preston [63] computed the curvature for the flow pictured on the disk in the left side of figure 4.1 and in Washabaugh-Preston [82] we computed the curvature on the 3D volumorphism group  $\mathcal{D}_\mu(M)$  where  $M$  is the solid flat torus. This 3D flow looks like the 2D flow, but stacked on itself. Much to our surprise, while the 2D flow exhibited non-positive curvature, the corresponding flow exhibited **non-negative** curvature, very much so the opposite of what we expected. This result hints that the 3D volumorphism group is much more positively curved than previously thought. If our previous discussions are any indication, positive curvature generally seems strongly connected to blow-up, so this discovery could indeed have significant ramifications for our future understanding.

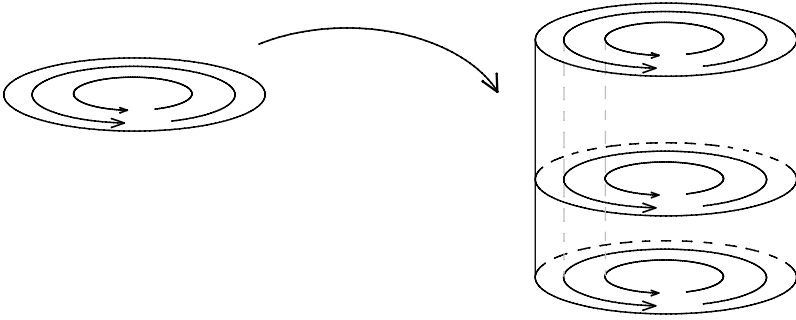


Figure 4.1: The flow whose curvature is computed in Washabaugh-Preston [82] is like the flow computed in Preston [63] but stacked on itself to form a cylinder. It's remarkable how similar these flows are, yet their associated curvatures are of opposite sign.

It's also worth mentioning that while axi-symmetric flow may seem different from the general vorticity model equation theme of this thesis, in fact as is discussed in Majda-Bertozzi [48], the axi-symmetric flow equations are fundamentally connected to the 2D Boussinesq equations. These in turn have significant connections to SQG. Along with the Wunsch equation, there really does exist the exciting possibility of understanding each of these equations associated geometric spaces as pieces of a more complete whole. Such a framework would make for a dramatic re-working of our understanding of these equations.

## 4.1 Introduction

Let  $(M, g)$  be a Riemannian manifold of dimension at least two with Riemannian volume form  $\mu$ . The configuration space for inviscid, incompressible fluid flows on  $M$  is the collection of volume-preserving diffeomorphisms (volumorphisms) of  $M$ , denoted by  $\mathcal{D}_\mu(M)$ . Arnold [1] demonstrated in 1966 that  $\mathcal{D}_\mu(M)$  can be thought of as an infinite dimensional Riemannian manifold. He also showed that flows obeying the Euler equations for inviscid, incompressible fluid flow can be realized as geodesics on  $\mathcal{D}_\mu(M)$ . This was proved rigorously in the context of Sobolev manifolds

by Ebin and Marsden [18]. Using this framework, questions of fluid mechanics can be re-phrased in terms of the Riemannian geometry of  $\mathcal{D}_\mu(M)$ . A good account of this is given in [2] or more recently in [38]. Of particular interest is the sectional curvature of  $\mathcal{D}_\mu(M)$ . As in finite dimensional geometry, given two geodesics with varying initial velocities in a region of strictly positive (resp. negative) sectional curvature, the two geodesics will converge (resp. diverge) via the Rauch Comparison theorem. In terms of fluid mechanics, this corresponds to the Lagrangian stability (resp. instability) of the associated fluid flows.

Arnold showed that the sectional curvature  $K(X, Y)$  of the plane in  $T_{\text{id}}\mathcal{D}_\mu(M)$  spanned by  $X$  and  $Y$  is often negative but occasionally positive. Rouchon [69] sharpened this to show that if  $M \subset \mathbb{R}^3$ , then  $K(X, Y) \geq 0$  for every  $Y \in T_{\text{id}}\mathcal{D}_\mu(M)$  if and only if  $X$  is a Killing field (i.e., one for which the flow generates a family of isometries). This result was generalized by Misiołek [51] and by Preston [62] for any manifold with  $\dim M \geq 2$ . This gives the impression that, in general,  $\mathcal{D}_\mu(M)$  will mostly be negatively curved. The question of when one can expect a divergence free vector field to give nonpositive sectional curvature remains open. However, Preston [63] provided criteria for divergence free vector fields of the form  $X = u(r)\partial_\theta$  on the area-preserving diffeomorphism groups of a rotationally-symmetric surface for which the sectional curvature  $K(X, Y)$  is nonpositive for all  $Y$ .

Our goal in this paper is to extend the curvature computation to  $\mathcal{D}_{\mu, E}(M)$ , the group of volume-preserving diffeomorphisms commuting with the flow of a Killing field  $E$ . In particular, we consider the solid flat torus,  $M = D^2 \times S^1$ , where  $D^2$  is the unit disk in  $\mathbb{R}^2$  and  $S^1$  is the unit circle, as a subset of  $\mathbb{R}^3$  with the planes  $z = 0$  and  $z = 2\pi$  identified, where  $E = \partial_\theta$  is the field corresponding to rotation in the disc. One may imagine fluid flows on this manifold as axisymmetric ideal flows with swirl on the solid infinite cylinder which are  $2\pi$ -periodic in the  $z$ -direction. We consider steady fluid velocity fields of the form  $X = u(r)\partial_\theta$ . The submanifold  $\mathcal{D}_{\mu, E}(M)$  is a totally geodesic submanifold of  $\mathcal{D}_\mu(M)$ , corresponding to the fact that an ideal fluid which is initially independent of  $\theta$  will always remain so. Hence we compute sectional curvatures  $K(X, Y)$  where  $Y \in T_{\text{id}}\mathcal{D}_{\mu, E}(M)$  is divergence-free **and** axisymmetric, i.e.,  $[E, Y] = 0$ .

In [63] it was shown that when  $X$  was considered as an element of  $\mathcal{D}_{\mu,F}(M)$  where  $F = \frac{\partial}{\partial z}$  (corresponding to considering  $X$  as a two-dimensional flow rather than a three-dimensional flow), the sectional curvature satisfied  $K(X, Y) \leq 0$  for every  $Y \in T_{\text{id}}\mathcal{D}_{\mu,F}(M)$  regardless of  $u(r)$ . By contrast we show here that if  $u$  satisfies the condition

$$\frac{d}{dr}(ru(r)^2) > 0, \quad (4.1)$$

then  $K(X, Y) > 0$  for every  $Y \in T_{\text{id}}\mathcal{D}_{\mu,E}(M)$ . We will also show that  $\frac{d}{dr}(ru(r)^2) \geq 0$  implies that  $K(X, Y) \geq 0$ . This does not contradict the result of Rouchon, since the proof of that result relies on being able to construct a divergence-free velocity field with small support which points in a given direction and is orthogonal to another direction, and there are not enough divergence-free vector fields in the axisymmetric case to accomplish this here.

The fact that the curvature is strictly positive in every section containing  $X$  makes it natural to ask whether there are conjugate points along every such corresponding geodesic. Unfortunately the Rauch comparison theorem cannot be used here, since  $\inf_{Y \in T_{\text{id}}\mathcal{D}_{\mu,E}(M)} K(X, Y) = 0$  even if (4.1) holds. Nonetheless we can show that as long as

$$ru(r)u'(r) + 2u(r)^2 > 0, \quad (4.2)$$

the geodesic formed by  $X = u(r)\partial_\theta$  has infinitely many monoconjugate points. It is easy to see that condition (4.1) implies (4.2). We do this by solving the Jacobi equation explicitly. As in [20], where the case  $u(r) = 1$  was considered, we can prove that these monoconjugate points have an epiconjugate point as a limit point, so that the differential of the exponential map is not Fredholm.

## 4.2 The Formula for Curvature

We first compute the curvature of  $\mathcal{D}_{\mu,E}(M)$  by expanding in a Fourier series in  $z$ . Notice first of all that any vector field  $Y$  which is tangent to  $\mathcal{D}_{\mu,E}(M)$  at the identity must be divergence-free and must commute with  $E = \frac{\partial}{\partial \theta}$ . Therefore we can write in the form

$$Y(r, z) = -\frac{g_z(r, z)}{r} \partial_r + \frac{g_r(r, z)}{r} \partial_z + f(r, z) \partial_\theta, \quad (4.3)$$

where  $f(0, z) = g(0, z) = 0$  and  $g(1, z)$  is constant in  $z$  (in order to be well-defined on the axis of symmetry and to have  $Y$  tangent to the boundary  $r = 1$ ). We think of the term  $-\frac{gz}{r}\partial_r + \frac{gr}{r}\partial_z$  as an analogue of the skew-gradient in two dimensions. We may express  $Y$  in a Fourier series in  $z$  as  $Y(r, z) = \sum_{n \in \mathbb{Z}} Y_n(r, z)$  where

$$Y_n(r, z) = e^{inz} \left[ -\frac{in}{r} g_n(r) \partial_r + \frac{g'_n(r)}{r} \partial_z + f_n(r) \partial_\theta \right]. \quad (4.4)$$

On any Riemannian manifold  $(M, g)$  with volume form  $\mu$ , a formula for the curvature tensor on  $\mathcal{D}_\mu(M)$  is given by

$$R(Y, X)X = P(\nabla_Y P(\nabla_X X) - \nabla_X P(\nabla_Y X) + \nabla_{[X, Y]} X), \quad (4.5)$$

where  $P(X)$  is the projection onto the divergence-free part of  $X$ . Concretely,  $P(X)$  is obtained by solving the Neumann boundary value problem

$$\begin{cases} \Delta q = \operatorname{div} X & \text{in } M \\ \langle \nabla q, \vec{n} \rangle = \langle X, \vec{n} \rangle & \text{on } \partial M \end{cases}$$

for  $q$  and then setting  $P(X) = X - \nabla q$ . The non-normalized sectional curvature is then given by

$$\bar{K}(X, Y) = \langle\langle R(Y, X)X, \bar{Y} \rangle\rangle = \int_M \langle R(Y, X)X, \bar{Y} \rangle \mu. \quad (4.6)$$

See [51] for the derivation of the formula we use here. We first compute  $R(Y_n, X)X$ .

**Proposition 4.2.1.** *Let  $M = D^2 \times S^1$ . Suppose that  $X \in T_{id} \mathcal{D}_{\mu, E}(M)$  is defined by  $X = u(r) \partial_\theta$ , and let  $Y_n$  be of the form (4.4). Then the curvature tensor  $R(Y_n, X)X$  is given by*

$$R(Y_n, X)X = P \left( -inug_n \left( 2u' + \frac{u}{r} \right) e^{inz} \partial_r + \frac{(q'_n + r f_n u) u e^{inz}}{r} \partial_\theta \right), \quad (4.7)$$

where  $q_n$  is the solution of the ODE

$$\begin{cases} \frac{1}{r} \frac{d}{dr} \left( r \frac{dq_n}{dr} \right) - n^2 q_n(r) = -\frac{1}{r} \frac{d}{dr} (r^2 f_n(r) u(r)) \text{ for } 0 < r < 1 \\ q'_n(1) = -f_n(1) u(1) \\ |q_n(0)| < \infty \end{cases} \quad (4.8)$$

*Proof.* We compute using formula (4.5). First note that  $\nabla_X X = -ru^2 \partial_r$ , which is the gradient of a function. Thus  $P(\nabla_X X) = 0$ . Next for  $n = 0$  note that

$$Y_0 = \frac{1}{r} g'_0(r) \partial_z + f_0(r) \partial_\theta$$

and  $\nabla_{Y_0} X = -rf_0(r)u(r)\partial_r$ .

This is also the gradient of a function, and thus

$$P(\nabla_{Y_0} X) = 0.$$

Now for  $n \neq 0$ ,

$$\nabla_{Y_n} X = -rf_n u e^{inz} \partial_r - \frac{in}{r} g_n \left(u' + \frac{u}{r}\right) e^{inz} \partial_\theta.$$

The solution  $q_n(r)e^{inz}$  of

$$\begin{cases} \Delta(q_n(r)e^{inz}) = \operatorname{div}(\nabla_{Y_n} X) & \text{in } M, \\ \langle \nabla(q_n e^{inz}), \vec{n} \rangle|_{\partial M} = \langle \nabla_{Y_n} X, \vec{n} \rangle|_{\partial M} & \text{on } \partial M, \end{cases} \quad (4.9)$$

clearly must satisfy (4.8). With this solution in hand we will get

$$\nabla_X(P(\nabla_{Y_n} X)) = inug_n \left(u' + \frac{u}{r}\right) e^{inz} \partial_r - \frac{(q'_n + rf_n u)ue^{inz}}{r} \partial_\theta.$$

We also easily compute

$$\nabla_{[X,Y]} X = -ing_n(r)u(r)u'(r)e^{inz} \partial_r.$$

So,  $R$  will be given by (4.7). □

To get a more explicit and useful formula for curvature, we proceed to solve the ODE (4.8).

**Lemma 4.2.1.** *If  $q_n$  satisfies (4.8), then the solution is given by*

$$q_n(r) = -\zeta_n(r)H_n(r) + \xi_n(r)J_n(r), \quad (4.10)$$

where

$$H_n(r) = \int_0^r s^2 f_n(s) u(s) \xi_n'(s) ds \quad \text{and} \quad J_n(s) = - \int_r^1 s^2 f_n(s) u(s) \zeta_n'(s) ds, \quad (4.11)$$

and

$$\xi_n(r) = I_0(nr) \quad \text{and} \quad \zeta_n(r) = \frac{K_1(n)}{I_1(n)} I_0(nr) + K_0(nr),$$

with  $I_0$  and  $K_0$  denoting the modified Bessel functions of the first and second kinds.

*Proof.* Since  $I_0$  and  $K_0$  solve the homogeneous version of (4.8), this is essentially just the variation of parameters formula together with an integration by parts. We simply verify the solution: taking the derivative of  $q_n(r)$ , we obtain

$$\begin{aligned} q_n'(r) &= -\zeta_n'(r)H_n(r) + \xi_n'(r)J_n(r) + r^2 f_n(r)u(r)(\xi_n(r)\zeta_n'(r) - \zeta_n(r)\xi_n'(r)) \\ &= -\zeta_n'(r)H_n(r) + \xi_n'(r)J_n(r) - r f_n(r)u(r), \end{aligned} \quad (4.12)$$

and since  $\zeta_n'(1) = J_n(1) = 0$  we get the correct boundary condition. Furthermore we get

$$q_n''(r) = -\zeta_n''(r)H_n(r) + \xi_n''(r)J_n(r) - \frac{d}{dr}(r f_n(r)u(r)),$$

and with these formulas we easily check that  $q_n$  satisfies (4.8).  $\square$

Plugging in the formula for  $q_n$  from Lemma 4.2.1 to the formula from Proposition 4.2.1, we obtain a very simple result.

**Theorem 28.** *On  $M = D^2 \times S^1$  with  $X = u(r) \partial_\theta$  and  $Y$  expressed as in (4.3), the non-normalized sectional curvature is given by  $\overline{K}(X, Y) = \sum_{n \in \mathbb{Z}} \overline{K}(X, Y_n)$ , where  $Y_n$  is expressed as in (4.4) and*

$$\overline{K}(X, Y_n) = 4\pi^2 \int_0^1 \frac{1}{r} \left( n^2 |g_n(r)|^2 \frac{d}{dr}(ru(r)^2) + \frac{|H_n(r)|^2}{I_1(nr)^2} \right) dr. \quad (4.13)$$

Hence the curvature is positive for all  $Y$  if and only if  $\frac{d}{dr}(ru(r)^2) > 0$ .

*Proof.* Using formula (4.12) in (4.7), we obtain

$$R(Y_n, X)X = P \left[ \left( -inug_n \left( 2u' + \frac{u}{r} \right) e^{inz} \right) \partial_r + \frac{u(\xi_n' J_n - \zeta_n' H_n) e^{inz}}{r} \partial_\theta \right],$$

which can clearly be expressed as  $e^{inz}$  times a function of  $r$  only. Orthogonality of the functions  $e^{imz}$  and  $e^{inz}$  over  $S^1$  when  $m \neq n$  implies that

$$\overline{K}(X, Y) = \sum_{m, n \in \mathbb{Z}} \langle\langle Y_m, R(Y_n, X)X \rangle\rangle = \sum_{n \in \mathbb{Z}} \langle\langle \overline{Y}_n, R(Y_n, X)X \rangle\rangle = \sum_{n \in \mathbb{Z}} \overline{K}(X, Y_n).$$

The latter is now relatively easy to compute. We have

$$\overline{K}(X, Y_n) = 4\pi^2 \int_0^1 \frac{n^2}{r^2} |g_n(r)|^2 \eta(r) dr + 4\pi^2 \int_0^1 r^2 \overline{f_n(r)} \left( u(r) (\xi'_n(r) J_n(r) - \zeta'(r) H_n(r)) \right) dr, \quad (4.14)$$

where  $\eta(r) = \frac{d}{dr}(ru(r)^2)$ . By the definitions (4.11) of  $H_n$  and  $J_n$ , we see that the second term in (4.14) is

$$4\pi^2 \int_0^1 \left( \overline{H'_n(r)} J_n(r) - \overline{J'_n(r)} H_n(r) \right) dr.$$

From here we adapt the corresponding computation in [64]. Integrating by parts and using the fact that  $J_n(r) \overline{H_n(r)} \rightarrow 0$  as  $r \rightarrow 0$  or  $r \rightarrow 1$ , we get

$$\int_0^1 \overline{H'_n(r)} J_n(r) - \overline{J'_n(r)} H_n(r) dr = -2\operatorname{Re} \int_0^1 \overline{J'_n(r)} H_n(r) dr = \int_0^1 \frac{J'_n(r)}{H'_n(r)} \frac{d}{dr} \left( |H_n(r)|^2 \right) dr,$$

and another integration by parts (where again the boundary terms vanish) gives

$$\int_0^1 \overline{H'_n(r)} J_n(r) - \overline{J'_n(r)} H_n(r) dr = - \int_0^1 \frac{d}{dr} \left( \frac{K_1(nr)}{I_1(nr)} \right) |H_n(r)|^2 dr.$$

Finally the Bessel function identity  $\frac{d}{dr} \left( \frac{K_1(r)}{I_1(r)} \right) = \frac{1}{rI_1(r)^2}$  implies (4.13).  $\square$

**Remark.** The normalized sectional curvature is given by  $K(X, Y) = \frac{\overline{K}}{\langle\langle X, X \rangle\rangle \langle\langle Y, Y \rangle\rangle - \langle\langle X, Y \rangle\rangle^2}$ . Suppose that  $f = 0$  and that only one  $g_n$  is nonzero in (4.4); then we have  $\langle\langle X, Y \rangle\rangle = 0$  and the sectional curvature takes the form

$$K(X, Y) = \frac{n^2 \int_0^1 \frac{1}{r} |g_n(r)|^2 \frac{d}{dr}(ru(r)^2) dr}{\left( \int_0^1 r^3 u(r)^2 dr \right) \left( \int_0^1 \left( \frac{n^2}{r} |g_n(r)|^2 + |g'_n(r)|^2 \right) dr \right)}.$$

We can make this arbitrarily small by choosing a highly oscillatory  $g_n$ . Hence although the curvature is strictly positive if  $\frac{d}{dr}(ru(r)^2) > 0$ , it cannot be bounded below by any positive constant.

### 4.3 Solution of the Jacobi equation

It is natural to ask whether the positive curvature guaranteed by the theorem above ensures the existence of conjugate points along the corresponding geodesic. This is not automatic since although the sectional curvature is positive in all sections containing the geodesic's tangent vector, it is not bounded below by any positive constant because of Remark 4.2; hence the Rauch comparison theorem cannot be applied directly. In this section we answer this question affirmatively by solving the Jacobi equation more or less explicitly along such a geodesic, and show that in fact conjugate points occur rather frequently.

**Theorem 29.** *Let  $\eta(t)$  be a geodesic on  $\mathcal{D}_{\mu,E}(D^2 \times S^1)$  with initial condition  $\eta(0) = id$  and  $\dot{\eta}(0) = X = u(r)\partial_\theta$ . Let  $\omega(r) = 2u(r) + ru'(r)$  denote the vorticity function of  $X$ , and assume that  $u(r)\omega(r) > 0$  for all  $r \in [0, 1]$ . Then  $\eta(t)$  is a monoconjugate point to  $\eta(0)$  for every time  $t = 2\pi\lambda/n$ , where  $n \in \mathbb{N}$  is arbitrary and  $\lambda$  is any eigenvalue of the Bessel-type Sturm-Liouville problem*

$$\frac{1}{r} \frac{d}{dr} (r\psi'(r)) - \left(n^2 + \frac{1}{r^2}\right)\psi(r) = -2\lambda^2 u(r)\omega(r)\psi(r), \quad \psi(1) = 0, \quad \psi(0) \text{ finite.}$$

*Proof.* Along a geodesic  $\eta(t)$  with (steady) Eulerian velocity field  $X$ , the Jacobi equation for a Jacobi field  $J(t) = Y(t) \circ \eta(t)$  may be written [62] as the system

$$\frac{\partial Y}{\partial t} + [X, Y(t)] = Z(t) \tag{4.15}$$

$$\frac{\partial Z}{\partial t} + P(\nabla_X Z(t) + \nabla_{Z(t)} X) = 0, \tag{4.16}$$

where  $P$  is the orthogonal projection onto divergence-free vector fields. The first equation is the linearized flow equation, while the second is the linearized Euler equation used in stability analysis.

Write

$$Z(t, r, z) = -\frac{1}{r} \frac{\partial h}{\partial z}(t, r, z) \partial_r + \frac{1}{r} \frac{\partial h}{\partial r}(t, r, z) \partial_z + j(t, r, z) \partial_\theta,$$

where  $h = 0$  on the axis  $r = 0$  and  $h$  is constant on the boundary  $r = 1$ . Then it is easy to compute

that (4.16) becomes the system

$$\frac{\partial j}{\partial t}(t, r, z) = \frac{\omega(r)}{r^2} \frac{\partial h}{\partial z}(t, r, z), \quad (4.17)$$

$$-\frac{1}{r} \frac{\partial^2 h}{\partial t \partial z}(t, r, z) \partial_r + \frac{1}{r} \frac{\partial^2 h}{\partial t \partial r}(t, r, z) \partial_z = 2P(ru(r)j(t, r, z) \partial_r), \quad (4.18)$$

where  $\omega(r) = 2u(r) + ru'(r)$  is the vorticity defined by  $\text{curl } X = \omega(r) \partial_z$ . Applying the curl to both sides of equation (4.18) to eliminate the projection operator, we obtain

$$\frac{\partial}{\partial t} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial h}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 h}{\partial z^2} \right] = -2ru(r) \frac{\partial j}{\partial z}. \quad (4.19)$$

Differentiating (4.19) in time and substituting (4.17) we obtain the single equation

$$\frac{\partial^2}{\partial t^2} \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial h}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 h}{\partial z^2} \right] = -\frac{2u(r)\omega(r)}{r} \frac{\partial^2 h}{\partial z^2}. \quad (4.20)$$

Expand  $h$  in a Fourier series in  $z$  to get

$$h(t, r, z) = \sum_{n \in \mathbb{Z}} h_n(t, r) e^{inz}.$$

Then for each  $n$  we can solve the eigenvalue problem

$$\frac{d}{dr} \left( \frac{1}{r} \phi'(r) \right) - \frac{n^2}{r} \phi(r) = \frac{2Cu(r)\omega(r)}{r} \phi(r);$$

to make this look more familiar we set  $\phi(r) = r\psi(r)$  and obtain

$$\frac{1}{r} \frac{d}{dr} (r\psi'(r)) - \left( n^2 + \frac{1}{r^2} \right) \psi(r) = 2Cu(r)\omega(r)\psi(r),$$

which is a singular Sturm-Liouville problem analogous to the Bessel equation. We obtain a sequence of eigenfunctions  $\phi_{mn}(r)$  for  $m \in \mathbb{N}$ , with eigenvalues  $C_{mn}$ . We see that

$$2C \int_0^1 \frac{u(r)\omega(r)}{r} \phi(r)^2 dr = - \int_0^1 \frac{1}{r} \phi'(r)^2 dr - \int_0^1 \frac{n^2}{r} \phi(r)^2 dr,$$

so that if  $\omega(r)u(r) > 0$ , then  $C$  must be strictly negative; we write  $C = -\lambda_{mn}^2$  for the eigenfunction  $\phi_{mn}(r)$ . Expanding  $h_n(t, r)$  in a basis of such eigenfunctions as

$$h(t, r, z) = \sum_{n \in \mathbb{Z}} \sum_{m=1}^{\infty} h_{mn}(t) \phi_{mn}(r) e^{inz},$$

equation (4.20) becomes

$$-\lambda_{mn}^2 h_{mn}''(t) = n^2 h_{mn}(t),$$

which obviously has solutions

$$h_{mn}(t) = a_{mn} \cos\left(\frac{nt}{\lambda_{mn}}\right) + b_{mn} \sin\left(\frac{nt}{\lambda_{mn}}\right)$$

for some coefficients  $a_{mn}$  and  $b_{mn}$ .

Suppose  $a_{m,n} = a_{m,-n} = \frac{1}{2}$  for some  $(m, n)$  with  $n \neq 0$ , and that all other  $a$  are zero and that every  $b$  is zero, so that  $h(t, r, z) = \cos\left(\frac{nt}{\lambda_{mn}}\right) \phi_{mn}(r) \cos nz$ . Then by equation (4.19) we compute that

$$j(t, r, z) = -\frac{\lambda_{mn}\omega(r)}{r^2} \phi_{mn}(r) \sin nz \sin\left(\frac{nt}{\lambda_{mn}}\right).$$

To find the Jacobi fields, write  $Y$  in equation (4.15) as

$$Y(t, r, z) = -\frac{1}{r} \frac{\partial g}{\partial z}(t, r, z) \partial_r + \frac{1}{r} \frac{\partial g}{\partial r}(t, r, z) \partial_z + f(t, r, z) \partial_\theta.$$

We easily compute that  $X = u(r) \frac{\partial}{\partial \theta}$  gives  $[X, Y] = \frac{1}{r} \frac{\partial g}{\partial z} u'(r) \frac{\partial}{\partial \theta}$ , and thus equation (4.15) becomes in components

$$\begin{aligned} \frac{\partial g}{\partial t}(t, r, z) &= h(t, r, z) \\ \frac{\partial f}{\partial t}(t, r, z) + \frac{u'(r)}{r} \frac{\partial g}{\partial z}(t, r, z) &= j(t, r, z). \end{aligned}$$

With  $g(0, r, z) = f(0, r, z) = 0$ , we find that

$$\begin{aligned} g(t, r, z) &= \frac{\lambda_{mn}}{n} \cos nz \sin\left(\frac{nt}{\lambda_{mn}}\right) \phi_{mn}(r) \\ f(t, r, z) &= \frac{2\lambda_{mn}^2 u(r)}{nr^2} \sin nz \left( \cos\left(\frac{nt}{\lambda_{mn}}\right) - 1 \right) \phi_{mn}(r). \end{aligned}$$

Thus both  $f$  and  $g$  vanish when  $t = 0$  and when  $t = 2\pi\lambda_{mn}/n$ , so  $\eta(2\pi\lambda_{mn}/n)$  is monoconjugate to the identity along  $\eta$ . □

**Remark.** Using the Sturm comparison theorem we can estimate the spacing of the eigenvalues  $\lambda_{mn}$  and show that for fixed  $m$  the sequence  $\lambda_{mn}/n$  has a finite limit as  $n \rightarrow \infty$ . Just as in [20], this

*must be an epiconjugate point. Therefore the differential of the exponential map is not Fredholm along any geodesic of this form. It is worth noting that the reason the Jacobi equation is explicitly solvable in this case is because there is no “drift” term, so the total time derivative agrees with the partial time derivative, in the same way as in [20].*

It would be very interesting to generalize the curvature computation to fields of the form  $X = u(r) \sin z \partial_\theta$ , which is the initial velocity field of the Hou-Luo initial condition [47] that leads numerically to a blowup solution. We expect that the formula  $\int \overline{H'_n} J_n - \overline{J'_n} H_n$  which appears both here and in [63] is a typical feature of curvature formulas when computed correctly, although they doubtless become substantially more complicated.

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