ON THE 2-D ORIENTATION PRESERVING CRYSTALLOGRAPHIC GROUPS

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Abstract. We explore certain geometric properties of the orientation preserving discrete subgroups of the isometries of the Euclidean plane (better known as the two-dimensional crystallographic groups/wallpaper groups). The nilpotency, spherical growth functions, limit shapes, and sprawls of these groups are discussed.

1. Introduction

1.1. **Definitions.** The *word metric*, denoted by $|\cdot|$, is a function which takes a group element g and gives the length of its spelling in terms of the generators of the group. For instance, $|abcd| = 4$ if a, b, c, and d are generators of the group. We let S_n denote the sphere of radius n and define the *limit shape* as $L := \lim_{n\to\infty} \frac{1}{n} S_n$, given that this limit exists. The *L-norm*, denoted by $||\cdot||_L$, is the unique norm for which L is the unit sphere. The *sprawl* of a group is defined as $E(G, S) := \lim_{n\to\infty} \frac{1}{|S_n|^2} \sum_{x,y,\in S_n} \frac{1}{n} d(x, y)$. In other words, *sprawl* is the average distance between the points of each sphere in the group, normalized by the radius, as the spheres get large. [1]

1.1.1. *Note on notation.* When we discuss an arbitrary group, we denote it by pn where $n \in \{1, 2, 3, 4, 6\}$. Given a word g in a group, we let $\vec{g}$ denote the vector with the coordinates of g in the Cayley graph.

1.2. **Motivation.** The main goal of this paper is to explore the limit shapes of the 2d orientation preserving crystallographic groups. Bieberbach explored and created theory for crystallographic groups almost 100 years ago. Since then, there have been many developments in geometric group theory, including the idea of a limit shape. Whenever one understands the limit shape, one understands other aspects of the groups embedding within its ambient space. Furthermore, knowing the limit shape makes calculations of group statistics easier. Burago found limit shapes in $\mathbb{R}^n$, in particular $\mathbb{R}^2$. Pansu showed limit shapes exist in certain nilpotent groups, and Breulliard gave similar results for certain groups of polynomial growth.

Although the results for some groups are known, none have yet explored the limit shapes of crystallographic groups; they have slipped through the cracks. We focus our attention on the two-dimensional orientation-preserving crystallographic groups, sometimes called the wallpaper groups. Four of these groups are not nilpotent, and none of them meet the conditions of groups Breulliard explored. Although others have showed limit shapes exist for certain groups, we explicitly give the limit shapes with simple generating sets.

Furthermore, we explore the notion of sprawl in these crystallographic groups. Duchin, Lelievre, and Mooney introduced sprawl and calculated it for free abelian groups. We calculate sprawl for some groups which are not free abelian, and find some interesting results for two of our groups.

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2. General Results

2.1. **Nilpotency.** There is well-known theory for nilpotent groups. However, groups which are not nilpotent are less understood. Specifically, the limit shapes of non nilpotent groups are not known. Out of the five orientation-preserving crystallographic groups, only one is nilpotent. $p1$ is defined as the crystallographic group generated by two translations and their inverses. Since $p1$ is isomorphic to $\mathbb{Z}^2$, it is free abelian and therefore nilpotent. However, the other crystallographic groups include rotations. Once one adds nontrivial, finite order rotations to the group, nilpotency is lost.

Theorem 1. *Let G be a crystallographic group which is not free abelian. G is not nilpotent.*

Proof. We must show that the lower central series of G does not terminate in the trivial group. That is,

$$G_0 = G, G_1 = [G, G], G_2 = [G_1, G], ..., G_n = [G_{n-1}, G] \neq 1$$

for all n $\in \mathbb{N}$. To do so, we proceed by induction.

It follows from Bieberbachs Theorem that the subgroup of translations is finite index [2]. Therefore, there are infinitely many translations in G; in particular, there is one. Thus, G_0 is nontrivial. Since G is not free abelian, it contains finite order elements. Let $\vec{x}$ be a vector in $\mathbb{E}^2$ and let s be a finite order element of G; that is, s is either a rotation about a point or a reflection about a line. Without loss of generality, we can assume s fixes the origin. Since s acts on the plane linearly, we can think of it as a 2 by 2 matrix.

Furthermore, assume that $\alpha \in [G_{n-1}, G]$ such that $\alpha : \vec{x} \longrightarrow \vec{x} + \vec{A}$ is a nontrivial translation and $\vec{A}$ is not fixed by s. Then the function $\gamma : \vec{x} \longrightarrow s\alpha s^{-1}\alpha^{-1}(\vec{x})$ is an element of $[G_{n-1}, G]$. We have,

$$\begin{aligned}\gamma(\vec{x}) &= s\alpha s^{-1}\alpha^{-1}(\vec{x}) \\ &= s\alpha s^{-1}(\vec{x} - \vec{A}) \\ &= s([s^{-1}\vec{x} - s^{-1}\vec{A}] + \vec{A}) \\ &= \vec{x} - \vec{A} + s\vec{A} \\ &= \vec{x} + (s - I)\vec{A}\end{aligned}$$

where I is the identity matrix. Thus the map γ takes $\vec{x}$ and adds the vector $(s - I)\vec{A}$ to it. This is clearly a translation, and is trivial if and only if $s\vec{A} = \vec{A}$, which in turn is true if and only if $\vec{A} = \vec{0}$ or $s = I$. Since we assumed s does not fix $\vec{A}$, $s \neq I$. We also assumed that α is a nontrivial translation, so $\vec{A} \neq \vec{0}$. We conclude that γ is a nontrivial translation. Thus, $[G_n, G]$ is nontrivial for all $n \in N$ and G is not nilpotent.

□

2.2. **Limit Shape.** Throughout the paper, we discuss the limit shapes of various crystallographic groups. In order to find the limit shapes, we will need the powerful lemma that follows. Note that this lemma is not limited to the crystallographic groups, but works for all groups in general.

Lemma 2.1. *Let G be any group. If $||g||_L - c_1 \leq |g| \leq ||g||_L + c_2$ for all $g \in G$, then L is the limit shape of G.*

Proof. We will show that given $\epsilon > 0$, there is an M such that for $n > M$, $\frac{1}{n}S_n \subset N_\epsilon(L)$. Let $k = max\{c_1, c_2\}$, set $M = 2k/\epsilon$, and let $n > M$. Let $g \in G$ such that $\vec{g} \in N_k(nL)$. Then, $||g||_L - k \leq ||g||_L \leq ||g||_L + k$. From our claim, we also have $||g||_L - k \leq |g| \leq ||g||_L + k$. Therefore, $d_\mathbb{R}(|g|, ||g||_L) \leq 2k = M\epsilon < n\epsilon$. We have, $\frac{1}{n}d_\mathbb{R}(|g|, ||g||_L) < \epsilon$ and it follows that $\frac{\vec{g}}{n} \in N_\epsilon(L)$. Since we chose g arbitrarily, this is true for all words lying in $N_k(nL)$ and therefore $\frac{1}{n}S_n \subset N_\epsilon(L)$. Since this is true for arbitrarily small ϵ, L is the limit shape. □

Using our generators, we discovered that the limit shapes of the pn are invariant under rotations of order n. We give a proof here.

Theorem 2. *Let G be one of the pn with limit shape L. Then L is invariant under rotations of order n.*

Proof. Consider a sector of the group (call it sector A). We show that if one chooses some g ∈A, then upon rotating by a to get the element ga, then the word metrics and L-norms of the two words are a constant away from each other. We must prove the following two equalities:

$$|g| = |ga| \pm 1$$

$$||g||_L = ||ga||_L \pm C$$

Where $g \in G$ and a is a generator of G. The first equality is clearly true. Consider the second. Since we are assuming that L is the limit shape, we know that $||g||_L \leq |g| \leq ||g||_L + k$ and $||ga||_L \leq |ga| \leq ||ga||_L + k_0$ where $k, k_0 \in \mathbb{R}$. Thus $||g||_L \leq |ga| \pm 1 \leq ||ga||_L + k_0$ and $|ga| \pm 1 \leq ||g||_L + k$. So, $||ga||_L$ is within a constant of $|ga|$, and $||g||_L$ is within a constant of $|ga|$, thus $||ga||_L$ must be within a constant of $||g||_L$. Thus the second equality holds. Now, from the second equality, we can see that if we choose a group element g that is far from the center of the cayley graph, that $\frac{||ga||_L}{||g||_L}$ will approach 1. Thus if we rotate an element of G by one of the generators, the L-norm remains unchanged. Thus the limit shape must also remain unchanged. □

Furthermore, the conjecture below describes how to obtain the limit shape of any crystallographic group. The conjecture holds for the five orientation-preserving groups we explore in this paper and is a much more efficient way of finding the limit shape.

Conjecture 1. *Let $M = \{g \in pn : g\, is\, a\, generator\, of\, T\}$, and let $k = |g|$, where $g \in M$. Take* ConvexHull(M) *and rotate it by rotations of order n in pn. Then*

$$L = \partial\frac{1}{k}\,\text{ConvexHull}(A_0 \bigcup ... \bigcup A_{n-1}),$$

where A_i is the ith rotate by $\frac{360}{n}$ of ConvexHull(M)*, fixing its center.*

Although the conjecture holds for our groups, we are not certain it holds for all groups. Therefore, we find the limit shapes in a different way; we will use the lemma below. Although we only need this lemma to prove the limit shape for $p3$, it is true for pn in general.

From [3], we know the generators of the translations are the translations of shortest word length. We will call these generators b and c.

Lemma 2.2. *Let g be an element of a two-dimensional, orientation-preserving crystallographic group. Then $g = b^n c^m a$ where b and c are generators of the translations, and a is a word of finite length.*

Proof. Let G be one of the pn and let T be the subgroup of translations in G. Since b and c are generators of T, we can write any translation as $b^n c^m$. Since T is a finite index subgroup, each group element is a bounded distance from a translation. Therefore, there is some path in the Cayley graph from $b^n c^m$ to g. Thus, $g = b^n c^m a$ for translations b and c and some finite length word, a. □

2.3. **Sprawl.** The Distance Lemma below will be used in the calculation of the sprawls, and is left in a general form.

Lemma 2.3. *(Distance Lemma). Let L be the boundary of a centrally symmetric convex polygon and σ and σ^{-1} be opposite sides. Let λ be the L-length of σ. Then*

$$\text{avedist}(\sigma, \sigma) = \frac{\lambda}{3}$$

and

$$\text{avedist}(\sigma, \sigma^{-1}) = 2.$$

Proof. Let e_1 and e_2 be the endpoint vertices of σ, and consider the parameterization $f : [0, 1] \to \sigma$ given by

$$f(s) = se_1 + (1 - s)e_2.$$

For $s, t \in [0, 1]$, the vectors $f(t) - f(s)$ all lie on the same line passing through the origin. It follows that

$$\left\| f(s) - f(t) \right\|_L = \lambda |s - t|$$

and one can compute

$$\begin{aligned} \text{avedist}(\sigma, \sigma) &= \int_0^1 \int_0^1 \lambda |s - t| ds dt \\ &= \frac{\lambda}{3}. \end{aligned}$$

The endpoints of σ^{-1} are $-e_1$ and $-e_2$. Now let $g : [0, 1] \to \sigma^{-1}$ be the parameterization

$$g(t) = t(-e_1) + (1 - t)(-e_2).$$

Choose $s, t \in [0, 1]$, and consider

$$\begin{aligned} f(s) - g(t) &= (s + t)e_1 + (2 - s - t)e_2 \\ &= ue_1 + (2 - u)e_2 \end{aligned}$$

where $u = s + t$. This is linear in u, which ranges over $[0, 2]$. When $u = 0$, we get $2e_2$, and when $u = 2$, we get $2e_1$. So the image of this map is precisely the line segment $2\sigma \subset 2L$. This proves that

$$\left\| f(s) - g(t) \right\|_L = 2$$

for all s and t. □

2.4. **Table of Results.** We give our results below. Notice that $p1, p2$, and $p4$ share similar characteristics; $p3$ and $p6$ do as well.

Group	Nilpotent	Limit Shape	Spherical Growth Function	Measure	Sprawl
p1	Yes	square	uniform	cone	4/3
p2	No	square	uniform	cone	4/3
p3	No	hexagon	oscillating	oscillating	DNE
p4	No	square	uniform	cone	4/3
p6	No	hexagon*	oscillating	oscillating*	DNE*

*These are conjectured results for p6.

3. Proofs/Details:

3.1. p1, p2, and p4.

3.1.1. *Definitions and construction of the Cayley graphs.* Out of the five orientation-preserving wallpaper groups, $p1, p2$, and $p3$ are the simplest to understand. We introduce them here and move on to discuss the more interesting groups $p3$ and $p6$ in the two following sections.

$p1$ is the simplest of the wallpaper groups. Recall that it consists only of translations, and is isomorphic to $\mathbb{Z}^2$.

$p2$ is generated by four elements, each of which are rotations by 180 degrees about a point. We choose to rotate about the points $(0, \pm\frac{1}{2}), (\pm\frac{1}{2}, 0)$. To construct the Cayley graph, we follow the orbit of the center of the square with these points as vertices.

$p4$ is generated by three elements; two are rotations by 90 degrees about two distinct points, and one rotation by 180 degrees about another point. We rotate about the points $(0, \pm 1)$ by 90 degrees and the point $(1, 0)$ by 180 degrees. Again, we follow the orbit of the center of a square with vertices $(\pm 1, \pm 1)$ to generate the Cayley graph.

These groups are isometric to each other and $\mathbb{Z}^2$. Since they are isometric, their Cayley graphs are the same up to linear transformation. Recall that the Cayley graph of $\mathbb{Z}^2$ is a grid. We omit the Cayley graphs for these groups, but encourage the reader to construct them if needed.

3.1.2. *Limit Shapes.* Duchin, Lelievre, and Mooney found limit shapes for all free abelian groups with arbitrary generating sets. We give the proof in a specific case here.

Theorem 3. *The limit shape of p1 is given by* $L = \text{ConvexHull}(\{(\pm 1, 0), (0, \pm 1)\})$. *In other words, it is the square with vertices* $(\pm 1, 0), (0 \pm 1)$.

Proof. We let L denote the square with vertices $(\pm 1, 0), (0, \pm 1)$. By Lemma 3.1, we need to show that $||g||_L \leq |g| \leq ||g||_L + C$ for some positive constant C. We will actually prove a stronger statement: $|g| = ||g||_L$ for all $g \in p1$. This is equivalent to showing $||g||_L = n$ for $g \in S_n$ and for all $n \in \mathbb{N}$. We proceed by induction.

The only words in S_1 are the generators, which are on L (in fact, they are the vertices). Next, assume for $n = k$ that $||g||_L = k$, where $g \in S_k$. Then, $\vec{g} \in kL$, the kth dilate of L. More precisely, $\vec{g} \in l_1 \cup l_2 \cup l_3 \cup l_4$, where $l_j = k\sigma_j$ for $j \in \{1, 2, 3, 4\}$. By symmetry, it suffices to consider the case when $\vec{g} \in l_1$. We parameterize this line segments as

$$l_1 : \; k < u, 1 - u >, 0 \leq u \leq 1$$

So, $\vec{g} =< ku_0, k - ku_0 >$ for some $u_0 \in [0,1]$. If we apply s^{-1} or t^{-1} to g, then we do not increase distance. Therefore, we only need to consider applying s and t to g. Applying these, we obtain

$$\vec{gs} = < 1, 0 > + < ku_0, k - ku_0 >= < 1 + ku_0, k - ku_0 >$$
$$\vec{gt} = < 0, 1 > + < ku_0, k - ku_0 >= < ku_0, 1 + k - ku_0 >$$

Setting $v = u_0$, we observe that $\vec{sg} \in l_1$ and $\vec{tg} \in l_1$, where

$$l_1' : < kv + 1, k - kv >, \qquad 0 \le v \le 1$$
$$l_1'' : < kv, k - kv + 1 >, \qquad 0 \le v \le 1$$

Since $l_1' \cup l_1'' = (k+1)\sigma_1$, $\vec{sg} \in (k+1)\sigma_1$ and $\vec{tg} \in (k+1)\sigma_1$. Thus $|sg| = ||sg||_L = k+1$ and $|tg| = ||tg||_L = k+1$. □

Theorem 4. *The limit shape of $p2$ is the square with vertices $(\pm 1, 0), (0, \pm 1)$.*

Proof. The identity transformation $T : \mathbb{E}^2 \longrightarrow \mathbb{E}^2$ sends $p2$ to $\mathbb{Z}^2$ and edges in the Cayley graph of $p2$ to the same edges in the Cayley graph of $\mathbb{Z}^2$. Therefore, with this generating set, $p2$ will have the same limit shape as $\mathbb{Z}^2$. □

Theorem 5. *The limit shape of $p4$ is the square with vertices $(\pm 1, 0), (0, \pm 1)$.*

The proof of the limit shape of $p4$ is similar to the proof of the limit shape of $p2$.

3.1.3. *Spherical growth functions.*

Theorem 6. *The spherical growth functions for $p1$, $p2$, and $p4$ are identical and are given by $\sigma(0) = 1, \sigma(1) = 4$, and $\sigma(n) = 4n$, for all $n > 1$.*

Proof. We observe that $p1$, $p2$, and $p4$ are isometric under the word length metric. Therefore, they have the same spherical growth function. Furthermore, since they are all isometric to $\mathbb{Z}^2$, they have the same spherical growth function of $\mathbb{Z}^2$, which is known to be $\sigma(0) = 1$, $\sigma(1) = 4$, and $\sigma(n) = 4n$, for all $n > 1$. □

3.1.4. *Sprawls.* Since $p1$, $p2$, and $p4$ are isometric to $\mathbb{Z}^2$, the cone measure used for $\mathbb{Z}^2$ (see [1] for the definition and derivation of cone measure) works for these groups as well. It also follows that these groups will have the same sprawl.

Theorem 7. $E(p1) = E(p2) = E(p4) = 4/3$.

We will need the following lemma.

Lemma 3.1. *Let S be the square with vertices $(\pm 1, \pm 1)$ equipped with the cone measure. Then, $E(S) = 4/3$.*

Proof. Let σ_1 be the top side of S and label the other sides as σ_2 through σ_4 moving around the square clockwise. Let x $\in S$. Due to rotational symmetry, it suffices to assume x $\in \sigma_1$. Define $a_j :=$ avedist(σ_1, σ_j). We must find a_j for $j = 1, 2, 3, 4$. First note that the L-length of σ_1 is 2. By the Distance Lemma, $a_1 = 2/3$ and $a_3 = 2$. By symmetry, $a_2 = a_4$, so it is enough to find a_2. Let x_s and y_t be points on σ_1 and σ_2, respectively. Then we have the following parameterizations of σ_1 and σ_2:

$$\vec{x}_s = < s, 1 >, \qquad -1 \le s \le 1$$
$$\vec{y}_t = < 1, t >, \qquad -1 \le t \le 1$$

We observe that when $s \leq t$, the ray based at the origin containing the vector $\vec{x}_s - \vec{y}_t$ intersects σ_1. On the other hand, when $s \geq t$, the ray intersects σ_4. These two cases lead to a piecewise distance function given by:

$$d_L(x_s, y_t) = \begin{cases} 1 - s & : s \leq t \\ 1 - t & : s \geq t \end{cases}$$

[Note: Since the distance function must be a piecewise linear function of two variables, we know it is described by two planes. We use the fact that $d_L(x_0, y_0) =, d_L(x_0, y_1) =$, and $d_L(x_1, y_1) =$ to find the equation of this plane in terms of s and t.]

We integrate and divide by the area of the region of integration and obtain

$$a_3 = \frac{1}{4}\left(\int_{-1}^{1}\int_{s}^{1} 1 - s\ dt\ ds + \int_{-1}^{1}\int_{-1}^{s} 1 - t\ dt\ ds\right) = \frac{4}{3}.$$

Now we can calculate the sprawl:

$$\begin{aligned} E(S) =& \frac{1}{4}a_1 + \frac{1}{4}a_2 + \frac{1}{4}a_3 + \frac{1}{4}a_4 \\ =& \frac{1}{4}\left(\frac{2}{3} + \frac{4}{3} + 2 + \frac{4}{3}\right) \\ =& \frac{4}{3}. \end{aligned}$$

□

Proof of Thm 5. Recall that L-norms, and therefore sprawl, are invariant under linear transformations. Also recall that given any two parallelograms, there is a linear map between them. Let L be a limit shape of one of p1, p2, or p4 and apply the necessary linear map from S to L. Since the sprawl is preserved under this map, we have $E(L) = E(S)$. This completes the proof. □

3.2. **p3.**

3.2.1. *Definition and construction of the Cayley graph.* $p3$ is defined as the group of isometries of rotations about two fixed points by 120 degrees. We choose to rotate about the points $(0, 1/2)$, which we call α, and $(0, -1/2)$, which we call β. We call our generators s and t (and their inverses), which are counterclockwise rotations by 120 degrees about α and β, respectively.

Consider the equilateral triange with sides of length 1 and vertices $(0, \pm 1/2)$ and $(-\sqrt{3}/2, 0)$. We apply s and t to this triangle and follow the orbit of the point $(0, 0)$. By doing so, we can easily identify the translations while also constructing a manageable Cayley graph.

Figure 1 shows the Cayley graph embedded in the Euclidean plane. The coordinates of the points are:

$$\begin{array}{lllll} (4x, y) & for\ \ x \in \mathbb{Z} & and & y = 4k\sqrt{3} & with\ \ k \in \mathbb{Z} \\ (4x+2, y) & for\ \ x \in \mathbb{Z} & and & y = (4k+2)\sqrt{3} & with\ \ k \in \mathbb{Z} \\ (2x+1, y) & for\ \ x \in \mathbb{Z} & and & y = (2k+1)\sqrt{3} & with\ \ k \in \mathbb{Z} \end{array}$$

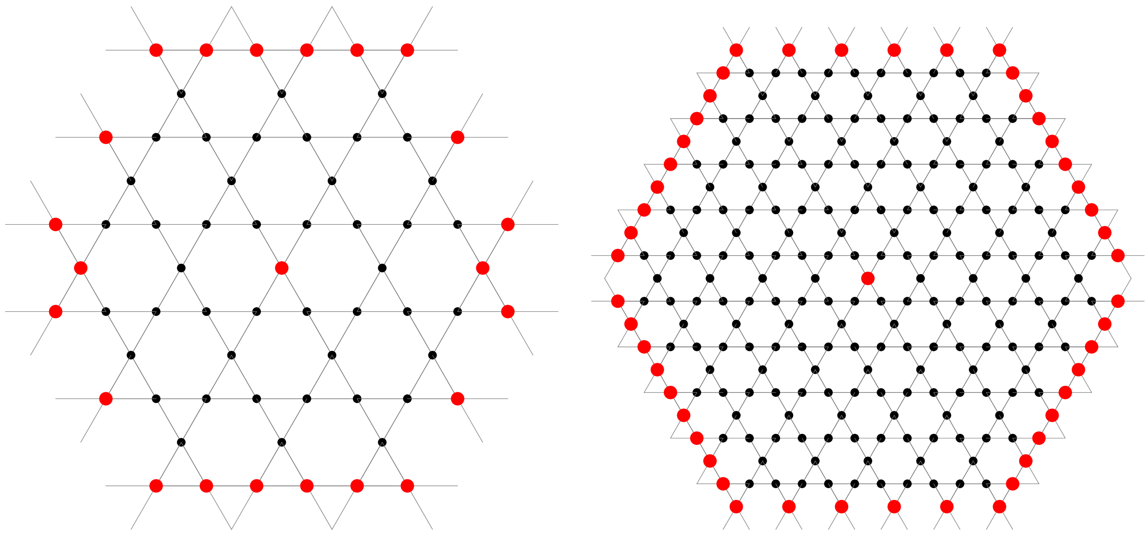

fig.1
Balls of radii 5 and 10 in p3

3.2.2. *Limit Shape.* The limit shape for p3 is a hexagon as shown below. The infinite cones of the hexagon are labeled A through F.

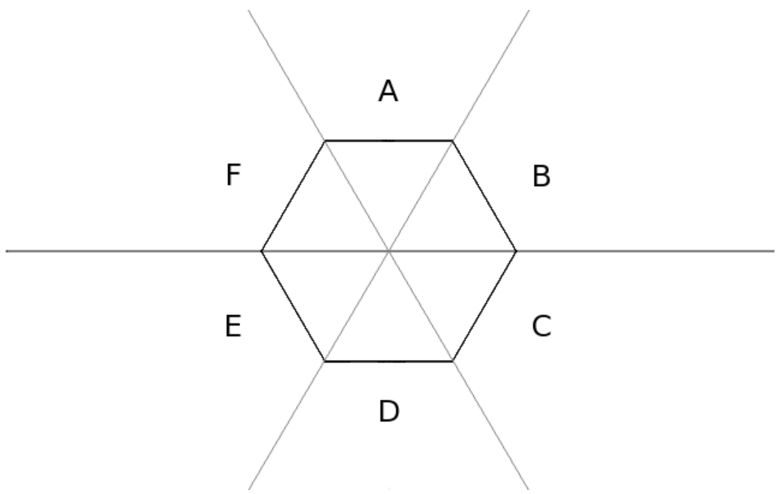

fig.2

Theorem 8. *For p3,* $L = \text{ConvexHull}[(\pm 1, \pm\sqrt{3}), (\pm 2, 0)]$. *In other words, the limit shape is the regular hexagon with vertices* $(\pm 1, \pm\sqrt{3})$, $(\pm 2, 0)$.

Lemma 3.2. *Let* $g \in p3$. *Then* $||g||_L \leq |g| \leq ||g||_L + 4$.

Proof. We let L denote the regular hexagon with vertices $(\pm 1, \pm\sqrt{3})$, $(\pm 2, 0)$. The first inequality is proved in [1].

For the second inequality, first consider $g \in A$. By Theorem 3, $g = b^n c^m a$ where b and c are two of the generators of the translations. Since b and c have word length 2, and since a is either the trivial rotation or a generator, we have

$$|g| = 2m + 2n + |a| \leq 2m + 2n + 1.$$

We now find $||g||_L$ in terms of n and m. Let $w \in A$ such that $w = b^n c^m$. In other words, $w = ga^{-1}$. The coordinates of $\vec{w}$ are

$$n < 2, 2\sqrt{3} > +m < -2, 2\sqrt{3} > = < 2(n-m), 2\sqrt{3}(n+m) > .$$

Looking at figure 3, we see that $||w||_L = |\vec{w}|/|\vec{v}|$. We notice σ_1 lies on the line $y_1 = \sqrt{3}$ and $\vec{w}$ lies on the line $y_2 = \sqrt{3}\left(\frac{n+m}{n-m}\right)x$.

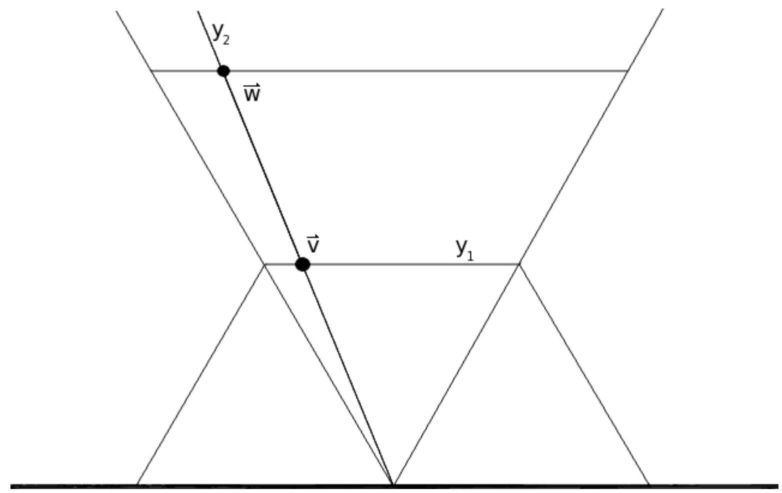

fig. 3

The intersection of y_1 and y_2 yields $\vec{v} =< \left(\frac{n-m}{n+m}\right), \sqrt{3} >$.
We have

$$||w||_L = \frac{|\vec{w}|}{|\vec{v}|} = \frac{\sqrt{[2(n-m)]^2 + [2\sqrt{3}(n+m)]^2}}{\sqrt{\left(\frac{n-m}{n+m}\right) + (\sqrt{3})^2}} = 2n + 2m$$

Since $w = ga^{-1}$, w is within one edge length on the Cayley graph from g. Since the length of any edge on the cayley graph is less than 1, it must be that $||g||_L \leq 2n + 2m + 1$. So, if $g \in A$,

$$||g||_L \leq |g| \leq 2n + 2m + 1 \leq ||g||_L + 2.$$

The proof for words in D is similar to that for words in A.

Now, let $q \in C$. One can easily show that cone C will lie inside of cone A after a 120^o rotation about point α. Thus when we apply s to q we find that $sq \in A$.

Since $sq \in A$,

$$||sq||_L \leq |sq| \leq ||sq||_L + 2$$

We know $|sq| \leq 2n + 2m + 1$, so it must be that

$$|q| = |s^{-1}sq| \leq 2n + 2m + 2$$

Now we find $||q||_L$. Let $\vec{B} =< 0, 1/2 >$, $\alpha : \vec{x} \to \vec{x} + \vec{B}$, and $\sigma = \begin{bmatrix} \cos 120^o & -\sin 120^o \\ \sin 120^o & \cos 120^o \end{bmatrix}$.
Note that σ rotates vectors counterclockwise about the origin by 120 degrees.

Since q=$s^{-1}sq$, we can reach $\vec{q}$ by rotating $\vec{sq}$ about the point α. We have

$$\vec{q} = \alpha^{-1}\sigma\alpha(\vec{sq}) = \alpha^{-1}\sigma(\vec{sq} + \vec{B}) = \sigma\vec{sq} + \sigma\vec{B} - \vec{B} = \sigma\vec{sq} + (\sigma - I)\vec{B}.$$

Since our candidate limit shape is invariant under rotations of 120 degrees about the origin, we know $||sq||_L = ||\sigma\vec{sq}||_L$. Furthermore, one can calculate that $|(\sigma - I)\vec{B}| = \frac{\sqrt{3}}{2}$. This implies:

$$||\sigma\vec{sq}||_L - |(\sigma - I)\vec{B}| \leq ||q||_L \leq ||\sigma(\vec{sq})||_L + |(\sigma - I)\vec{B}|$$

thus,

$$||\vec{sq}||_L - \frac{\sqrt{3}}{2} \leq ||q||_L \leq ||\vec{sq}||_L + \frac{\sqrt{3}}{2}$$

Therefore,

$$2n + 2m - 2 \leq ||q_L|| \leq 2n + 2m + 2$$

and finally,

$$|q| \leq 2n + 2m + 2 = 2n + 2m - 2 + 4 \leq ||q||_L + 4.$$

The proofs for the other words in the cones B, E, and F are similar to that for words in C. □

3.2.3. *Spherical Growth Function.* In Topics in Geometric Group Theory, Pierre de la Harpe gives an example of a group with a nonuniform spherical growth function. He describes this group as the group of orientation-preserving reflections of an equilateral triangle in the Euclidean plane. This group is in fact our crystallographic group $p3$! He gives the spherical growth function that follows.

Proposition 3.3. *The spherical growth function of* $p3$ *is given by* $\sigma(0) = 1,\ \sigma(1) = 4$ *and*

$$\sigma(2k-1) = 8k - 2 \qquad \textit{for all} \qquad k \geq 2$$
$$\sigma(2k) = 10k - 2 \qquad \textit{for all} \qquad k \geq 1.$$

This spherical growth function oscillates between even and odd spheres, which leads to interesting results. As we discuss in the next section, the measure also oscillates between even and odd spheres. In section 4.2.5., we show that $p3$ has no sprawl, which is the first discovery of such a group! We believe the absence of sprawl is due to this oscillating spherical growth function as it affects the measure used in the calculation.

3.2.4. *Limiting Measures.*

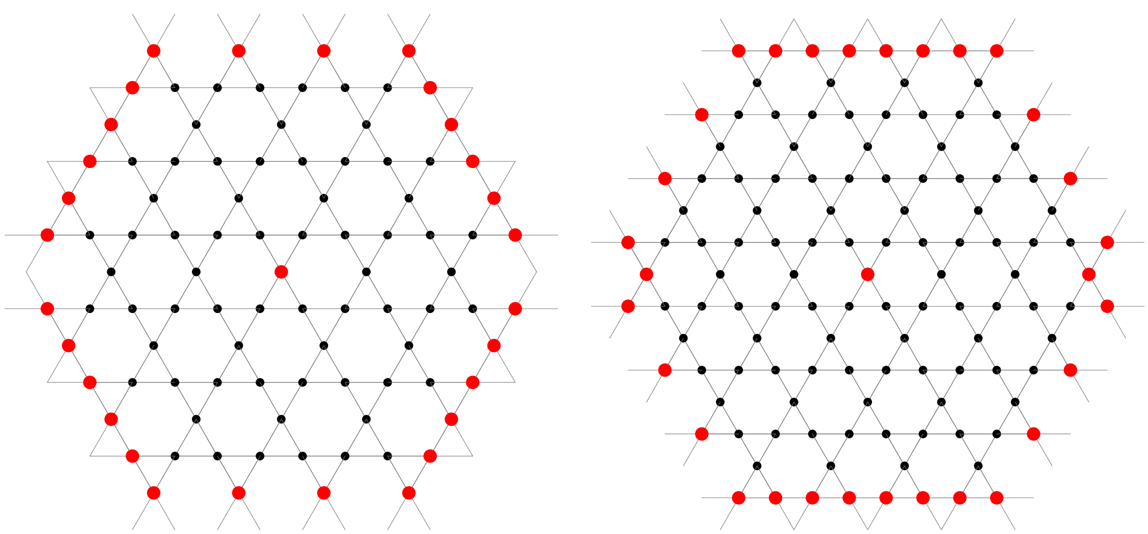

fig.4

Balls of radius 6 and 7, respectively, in P3

Before we compute the sprawls of the even and odd spheres, we must first find the counting measures. Note that in a given sphere, words are not uniformly distributed. Furthermore, the distribution of words in even spheres differs from that of odd spheres. It follows from these two observations that the cone measure will not work here (see [1] for the definition of the cone measure). We define

$$\mu_{even}(\sigma_i) := \lim_{k\to\infty} \frac{\#(\Delta\sigma_i \cap S_{2k})}{\#S_{2k}}$$
$$\mu_{odd}(\sigma_i) := \lim_{k\to\infty} \frac{\#(\Delta\sigma_i \cap S_{2k-1})}{\#S_{2k-1}}.$$

Looking at even spheres, we observe that

$$\#(\Delta\sigma_1 \cap S_{2k}) = \#(\Delta\sigma_4 \cap S_{2k}) = k+1$$

and

$$\#(\Delta\sigma_2 \cap S_{2k}) = \#(\Delta\sigma_3 \cap S_{2k}) = \#(\Delta\sigma_5 \cap S_{2k}) = \#(\Delta\sigma_6 \cap S_{2k}) = 2k.$$

From Proposition 4.3, $\#S_{2k} = 10k - 2$. We compute

$$\mu_{even}(\sigma_1) = \mu_{even}(\sigma_4) = \frac{1}{10}$$
$$\mu_{even}(\sigma_2) = \mu_{even}(\sigma_3) = \mu_{even}(\sigma_5) = \mu_{even}(\sigma_6) = \frac{1}{5}.$$

Sphere points are uniformly distributed on each side; therefore, given $\tau \subset L$, we have

$$\mu_{even}(\tau) = \sum_{i=1}^{6} \mu_{even}(\tau \cap \sigma_i).$$

Now we look at odd spheres and see that

$$\#(\Delta\sigma_1 \cap S_{2k-1}) = \#(\Delta\sigma_4 \cap S_{2k-1}) = 2k$$

and

$$\#(\Delta\sigma_2 \cap S_{2k-1}) = \#(\Delta\sigma_3 \cap S_{2k-1}) = \#(\Delta\sigma_5 \cap S_{2k-1}) = \#(\Delta\sigma_6 \cap S_{2k-1}) = k.$$

From Proposition 4.3, $\#S_{2k-1} = 8k - 2$, so we have

$$\mu_{odd}(\sigma_1) = \mu_{odd}(\sigma_4) = \frac{1}{4}$$
$$\mu_{odd}(\sigma_2) = \mu_{odd}(\sigma_3) = \mu_{odd}(\sigma_5) = \mu_{odd}(\sigma_6) = \frac{1}{8}.$$

Once again, we observe that sphere points are uniformly distributed on each side. So given $\tau \subset L$,

$$\mu_{odd}(\tau) = \sum_{i=1}^{6} \mu_{odd}(\tau \cap \sigma_i).$$

3.2.5. *Sprawls.*
We are now ready to compute sprawl for even and odd spheres. Let σ_1 be the top side of $T(L)$ and label the other sides as σ_2 through σ_6 moving around the hexagon clockwise. Furthermore, we define even and odd sprawl as follows:

$$E_{even}(L) := \int_{L\times L} ||\,\mathrm{x}-\mathrm{y}\,||_L \; d\mu_{even}(\mathrm{x})d\mu_{even}(\mathrm{y})$$
$$E_{odd}(L) := \int_{L\times L} ||\,\mathrm{x}-\mathrm{y}\,||_L \; d\mu_{odd}(\mathrm{x})d\mu_{odd}(\mathrm{y})$$

Theorem 9. $E_{even}(L) = 13/10$ *and* $E_{odd}(L) = 5/4$.

Proof. Let $\mathrm{x} \in L$. Due to rotational symmetry, it suffices to assume $\mathrm{x} \in \sigma_1$. Define $a_j := \mathrm{avedist}(\sigma_1, \sigma_j)$. We must find a_j for $j = 1, 2, ..., 6$. By symmetry, it is clear that $a_2 = a_6$ and $a_3 = a_5$. Therefore, it is only necessary to calculate a_j for $j = 1, 2, 3, 4$.

Since the L-length of σ_1 is 1, it follows from the Distance Lemma that $a_1 = \frac{1}{3}$ and $a_4 = 2$.

Next, we calculate a_2. Let x_s and y_t be points on σ_1 and σ_2, respectively. Then we have the following parameterizations of σ_1 and σ_2:

$$\vec{x}_s = < s, \sqrt{3} >, \qquad -1 \le s \le 1$$
$$\vec{y}_t = < t, 2\sqrt{3} - \sqrt{3}t >, \quad 1 \le t \le 2$$

We observe that the ray based at the origin containing the vector $\vec{x}_s - \vec{y}_t$ intersects σ_6, independent of s and t. Letting v be the point of intersection, we can compute the L-norm:

$$d_L(x_s, y_t) = \frac{d_2(x_s,\ y_t)}{d_2(v,\ 0)}$$

First, we find the coordinates of v. σ_1 is a segment of the line $y = 2\sqrt{3} + \sqrt{3}x$ and $\vec{x}_s - \vec{y}_t =< s-t, \sqrt{3}(t-1) >$ lies on the line $y = \sqrt{3}\left(\frac{t-1}{s-t}\right)x$. The coordinates of the point of interestection are

$$x = 2\left(\frac{s-t}{2t-s-1}\right), \quad y = 2\sqrt{3}\left(\frac{t-1}{2t-s-1}\right).$$

Furthermore,

$$d_2(v,0) = \sqrt{[2(\frac{s-t}{2t-s-1})]^2 + [2\sqrt{3}(\frac{t-1}{2t-s-1})]^2}$$
$$= \frac{2}{2t-s-1}\sqrt{(s-t)^2 + 3(t-1)^2}$$

and

$$\begin{aligned} d_2(x_s, y_t) =& \sqrt{(s-t)^2 + [\sqrt{3}(t-1)]^2} \\ =& \sqrt{(s-t)^2 + 3(t-1)^2}. \end{aligned}$$

We have,

$$d_L(x_s, y_t) = \frac{\sqrt{(s-t)^2 + 3(t-1)^2}}{\frac{2}{2t-s-1}\sqrt{(s-t)^2 + 3(t-1)^2}} = \frac{2t-s-1}{2}.$$

We now integrate this distance function over our parameters and divide by the area of the region of integration to find the average distance between σ_1 and σ_2:

$$a_2 = \frac{1}{2}\int_{-1}^{1}\int_{1}^{2} \frac{2t-s-1}{2}\, dt\, ds = 1.$$

The calculation for a_3 slightly more complicated, but similar to the calculation in Lemma 4.1. If we let x_s and z_r be points on σ_1 and σ_3, respectively, then the ray that contains $\vec{x}_s - \vec{z}_r$ does not always interestect the same side. There are two cases and the distance function is

$$d_L(x_s, z_r) = \begin{cases} 2-r & : r \leq s \\ 2-s & : r \geq s \end{cases}$$

We integrate our piecewise distance function to find a_3:

$$a_3 = \int_0^1 \int_0^s 2-r\ drds + \int_0^1 \int_s^1 2-s\ drds = \frac{5}{3}.$$

Finally, we compute

$$\begin{aligned} E_{even}(L) =& \frac{1}{10}a_1 + \frac{1}{5}a_2 + \frac{1}{5}a_3 + \frac{1}{10}a_4 + \frac{1}{5}a_5 + \frac{1}{5}a_6 \\ =& \frac{1}{10}(a_1 + a_4) + \frac{1}{5}(a_2 + a_3 + a_5 + a_6) \\ =& \frac{1}{10}(a_1 + a_4) + \frac{1}{5}(2a_2 + 2a_3) \\ =& \frac{1}{10}\left(\frac{1}{3} + 2\right) + \frac{1}{5}\left[2(1) + 2\left(\frac{5}{3}\right)\right] \\ =& \frac{13}{10} \end{aligned}$$

$$\begin{aligned} E_{odd}(L) =& \frac{1}{4}a_1 + \frac{1}{8}a_2 + \frac{1}{8}a_3 + \frac{1}{4}a_4 + \frac{1}{8}a_5 + \frac{1}{8}a_6 \\ =& \frac{1}{4}(a_1 + a_4) + \frac{1}{8}(a_2 + a_3 + a_5 + a_6) \\ =& \frac{1}{4}(a_1 + a_4) + \frac{1}{8}(2a_2 + 2a_3) \\ =& \frac{1}{4}\left(\frac{1}{3} + 2\right) + \frac{1}{8}\left[2(1) + 2\left(\frac{5}{3}\right)\right] \\ =& \frac{5}{4} \end{aligned}$$

□

Corollary 3.4. *The sprawl for even spheres is different than the sprawl for odd spheres. Thus, we have found a group without sprawl!*

Interestingly, $E_{even}(L) \in [4/\pi, 4/3]$ but $E_{odd}(L) \notin [4/\pi, 4/3]$, whereas it is conjectured that

$$\{E(\Omega)\} = [4/\pi, 4/3]$$

where Ω is the set of convex, centrally symmetric bodies in $\mathbb{R}^2$. [1]

3.3. **p6.**

3.3.1. *Definition and construction of the Cayley graph.* $p6$ is defined as the group of isometries of rotations about two fixed points by 60 degrees. We choose to rotate about the points $(-\frac{3}{8}, -\frac{5\sqrt{3}}{24})$, which we call B, and $(\frac{1}{8}, \frac{7\sqrt{3}}{24})$, which we call C. We choose to rotate about these points because they are equidistant from the origin and make the fundamental domain an isoscelese triangle. We call our generators s and t (and their inverses), which are counterclockwise rotations by 60 degrees about B and C, respectively.

Consider the fundamental domain; that is, the isoscelese triangle with hypontenuse of length 1 and vertices $(-\frac{3}{8}, -\frac{5\sqrt{3}}{24})$, $(\frac{1}{8}, \frac{7\sqrt{3}}{24})$, and $(\frac{1}{8}, -\frac{\sqrt{3}}{24})$. We apply s and t to this triangle and follow the orbit of the point $(0, 0)$. By doing so, we can easily identify the translations while also constructing a manageable Cayley graph.

Figure 5 shows the Cayley graph embedded in the Euclidean plane.

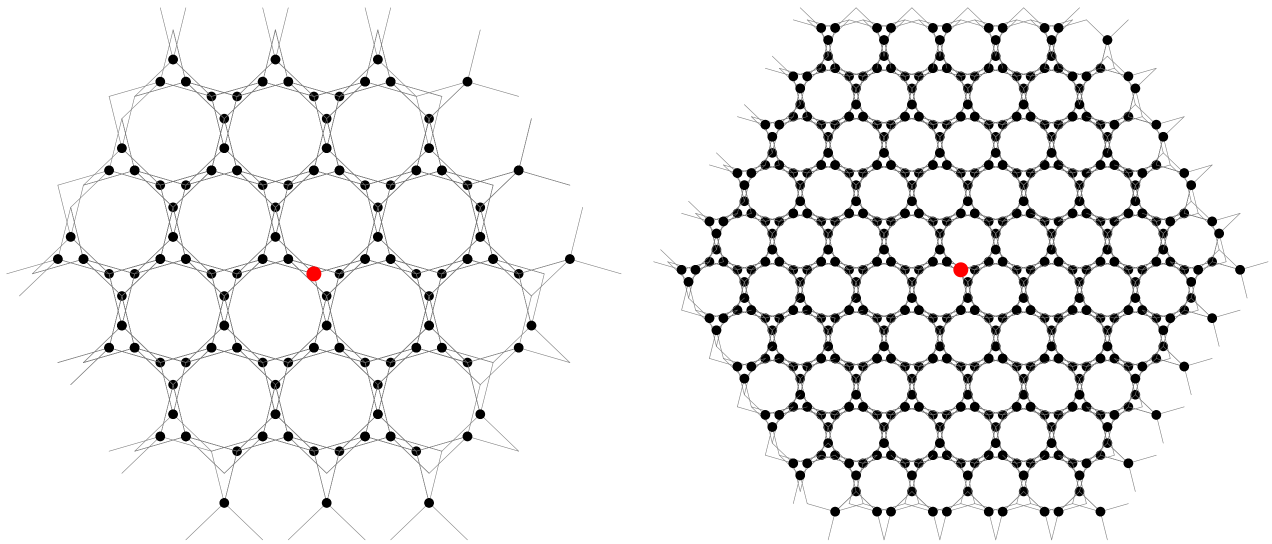

fig.5
Balls of radius 5 and 10 in p6

3.3.2. *Limit shape.* When looking at spheres of large radii, one can see that the limit shape appears to be a hexagon (see fig.5).

3.3.3. *Spherical growth function.*

Conjecture 2. *The spherical growth function for* $p6$ *is given by* $\sigma(0) = 1, \sigma(1) = 4, \sigma(2) = 12, \sigma(2k-1) = 18k - 14, \sigma(2k) = 18k - 4$ *for all* $k \geq 2$

This function works for all spheres we have tested, but may fail with larger spheres. Either way, it is clear that the spherical growth function is oscillating and not uniform.

3.3.4. *Measure and sprawl.* Due to the oscillating spherical growth function, and judging by the figures, it appears that p6 has an oscillating measure. It is clear that the cone measure will not work. This is similar to our results for $p3$; the measure oscillates between even and odd spheres. Since we do not have a nice measure, we conjecture that p6 will not have sprawl. If this is the case, then we have found two groups without sprawl!

4. Other Directions

We have thus classified the limit shapes of the five orientation-preserving, two-dimensional crystallographic groups with their natural generating sets. Although we omitted proofs of the properties of $p6$, we intend to research this group more and give detailed proofs in our next paper. We also intend to further explore the geometric properties of other crystallographic groups.

There are several natural ways to expand the work done in this paper. Of the 17 wallpaper groups, we have only looked at the five orientation-preserving groups. Although we know that none of these other groups are nilpotent, we do not know what effect non-orientation-preserving isometries (i.e. reflections and glide reflections) will induce on their limit shapes, or on the computation of their sprawls. For any group that has rotations, it seems that adding reflections would allow one to move around the group easier. Because sprawl characterizes the ability to move around the group, we may find that sprawl decreases with the addition of reflections.

It would also be useful to look at the effects of different (or even arbitrary) generating sets as is done in [1] for $\mathbb{Z}^2$. Perhaps each wallpaper group has a range of sprawls depending on their generators. Maybe one could find a generating set that gives $p3$ sprawl! Also, one may be able to generalize Picks Theorem to determine the limit shapes given by arbitrary generators (as is done in [1]).

Another route of study would be to attempt to understand how all of these properties generalize to higher dimensional crystallographic groups. For instance, there are 230 three-dimensional and 4,895 four-dimensional crystallographic groups [5]. It would be nice to generalize the methods used in this paper in order to classify the characteristics of these higher-dimensional groups as well.

5. References

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